Lecture note on 3+1 formalism of numerical relativity

Masaru Shibata

(Yukawa Institute, Kyoto U)

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Basis equations for Numerical Relativity

$$G_{\mu\nu} = 8\pi \frac{G}{c^4} T_{\mu\nu}$$

← **Einstein equation**

$$\begin{pmatrix} \nabla_\mu T^\mu_\nu = 0 \\ \nabla_\mu \left(\rho u^\mu\right) = 0 \\ \nabla_\mu \left(\rho u^\mu Y_l\right) = Q_l \\ \nabla_\mu F^{\mu\nu} = -4\pi j^\nu \\ \nabla_{[\mu} F_{\nu\lambda]} = 0 \\ p^\alpha \partial_\alpha f + \dot{p}^\alpha \frac{\partial f}{\partial p^\alpha} = S \end{pmatrix}$$

← **Matter fields:**

Next time if any

Others in Numerical Relativity

- Imposing gauge conditions
- Extracting gravitational waves
- Finding black holes
  (finding apparent horizon)
- Adaptive mesh refinement

Contents

1. Structure of Einstein's equation (briefly)
2. 3+1 formalism of Einstein's equation
3. BSSN formalism
4. Gauge conditions
5. Initial value problem
6. Implementation of finite differencing
7. Extracting gravitational waves
8. Adaptive mesh refinement
9. Apparent horizon finder

Solution of Einstein's equation for *dynamical phenomena*

- We have to solve Einstein's equations as an ***initial value problem***
- Einstein's equations, $G_{\mu\nu} = 8\pi G c^{-4} T_{\mu\nu}$, are *equations for space and time*
  → Space and time coordinates appear in a mixed way; *"time coordinate" does not always have the property of time.*

E.g., Schwarzschild coordinates;
t is time for $r > 2M$, but is not for $r < 2M$

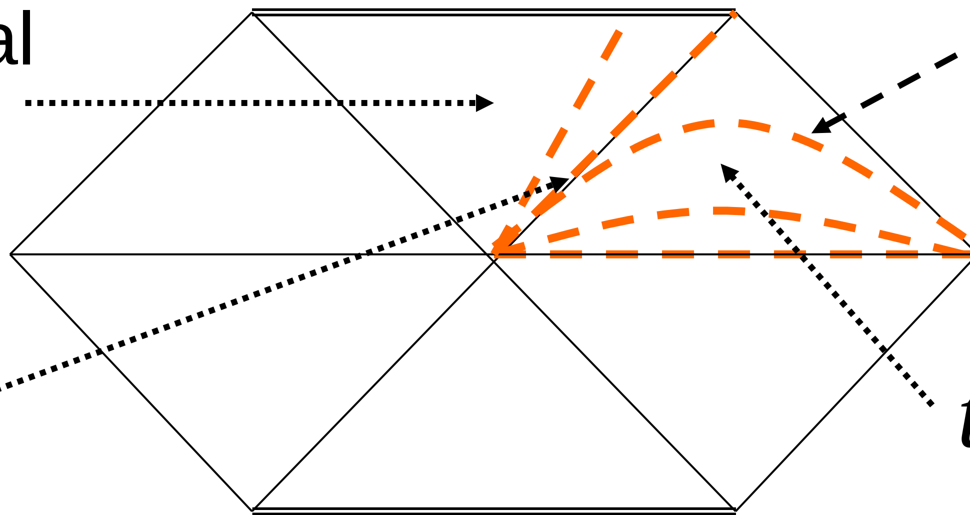
→ **Special formalism is necessary to follow dynamics of a variety of spacetimes**

Schwarzschild spacetime

$$ds^2 = -\left(c^2 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

t is a radial coordinate

$t = \text{const}$



Coordinate singularity

t is the time coordinate.

Time has to be always "time" in numerical relativity

Several formulations

1. **3+1 (*N*+1) formalism**
2. **Formulation based on a special (harmonic) coordinates**
   (e.g., Pretorius formalism; also often used in Post-Newtonian theory)
3. **Hyperbolic formalism**
   (e.g., Kidder-Scheel-Teukolsky formalism)

- Others …..

In this lecture, I focus on the first one

Einstein's equation = hyperbolic equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \frac{G}{c^4} T_{\mu\nu}$$

$$\begin{cases} G_{\mu\nu} : \text{Einstein tensor} \\ R_{\mu\nu} : \text{Ricci tensor} \end{cases}$$

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\nu\alpha} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu}$$

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right)$$

$$\Rightarrow \quad (-g) G^{\mu\nu} = \partial_\alpha \partial_\beta \left[(-g)\left(g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta} \right) \right] + (-g) t^{\mu\nu}$$

$t^{\mu\nu}$: Pseudo tensor of Landau-Lifshitz $O\left[\left(\partial g_{\mu\nu} \right)^2 \right]$

de-Donder Gauge: $\partial_\alpha \left(\sqrt{-g}\, g^{\alpha\beta} \right) = 0$

Wave equation

$$\Rightarrow \quad G^{\mu\nu} = \sqrt{-g}\, g^{\alpha\beta} \partial_\alpha \partial_\beta \left(\sqrt{-g}\, g^{\mu\nu} \right) + O\left[\left(\partial g_{\mu\nu} \right)^2 \right]$$

Einstein's equation is similar to scalar wave equation

① $\partial_t^2 \phi = \Delta \phi$ ⟵ Post-Newtonian way

② $\Rightarrow \begin{Bmatrix} \eta \equiv \partial_t \phi \\ \partial_t \eta = \Delta \phi \end{Bmatrix}$ ⟵ <u>**3+1 way**</u>

$(\eta, \phi) \leftrightarrow (K_{ij}, \gamma_{ij})$

or

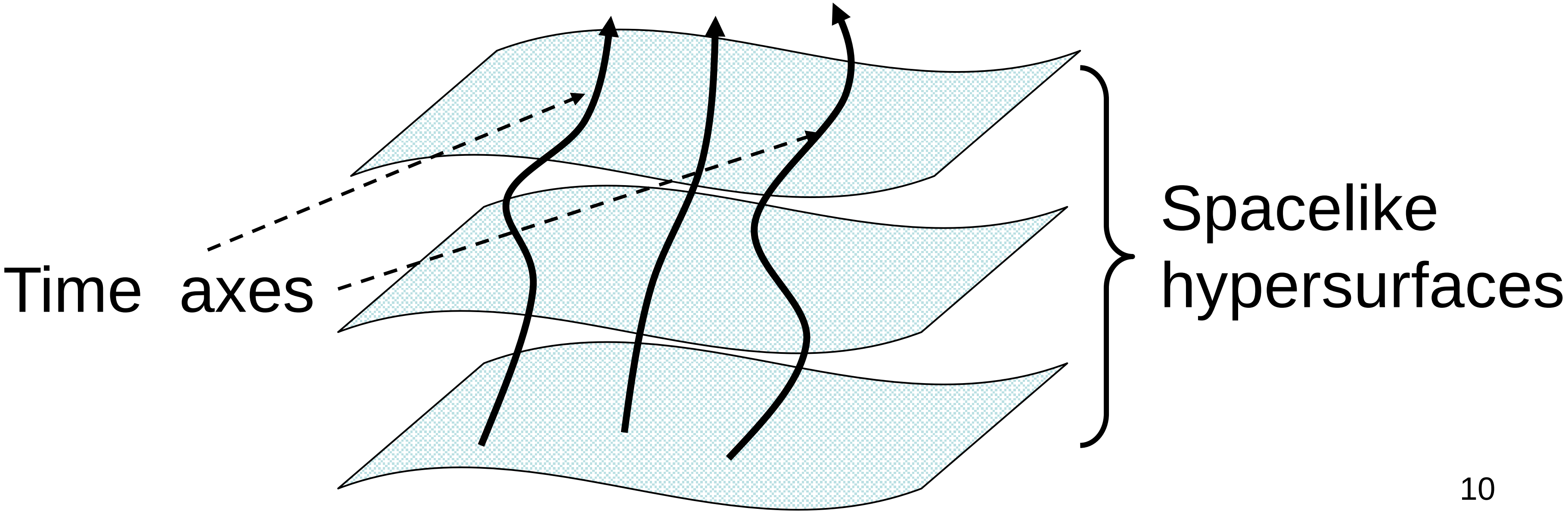
③ $\Rightarrow \begin{Bmatrix} \partial_t \zeta_i = \partial_i \phi \\ \partial_t \phi = \partial_i \zeta^i \end{Bmatrix}$ ⟵ Hyperbolic way

Similar: However, spacetime is not a priori given in general relativity

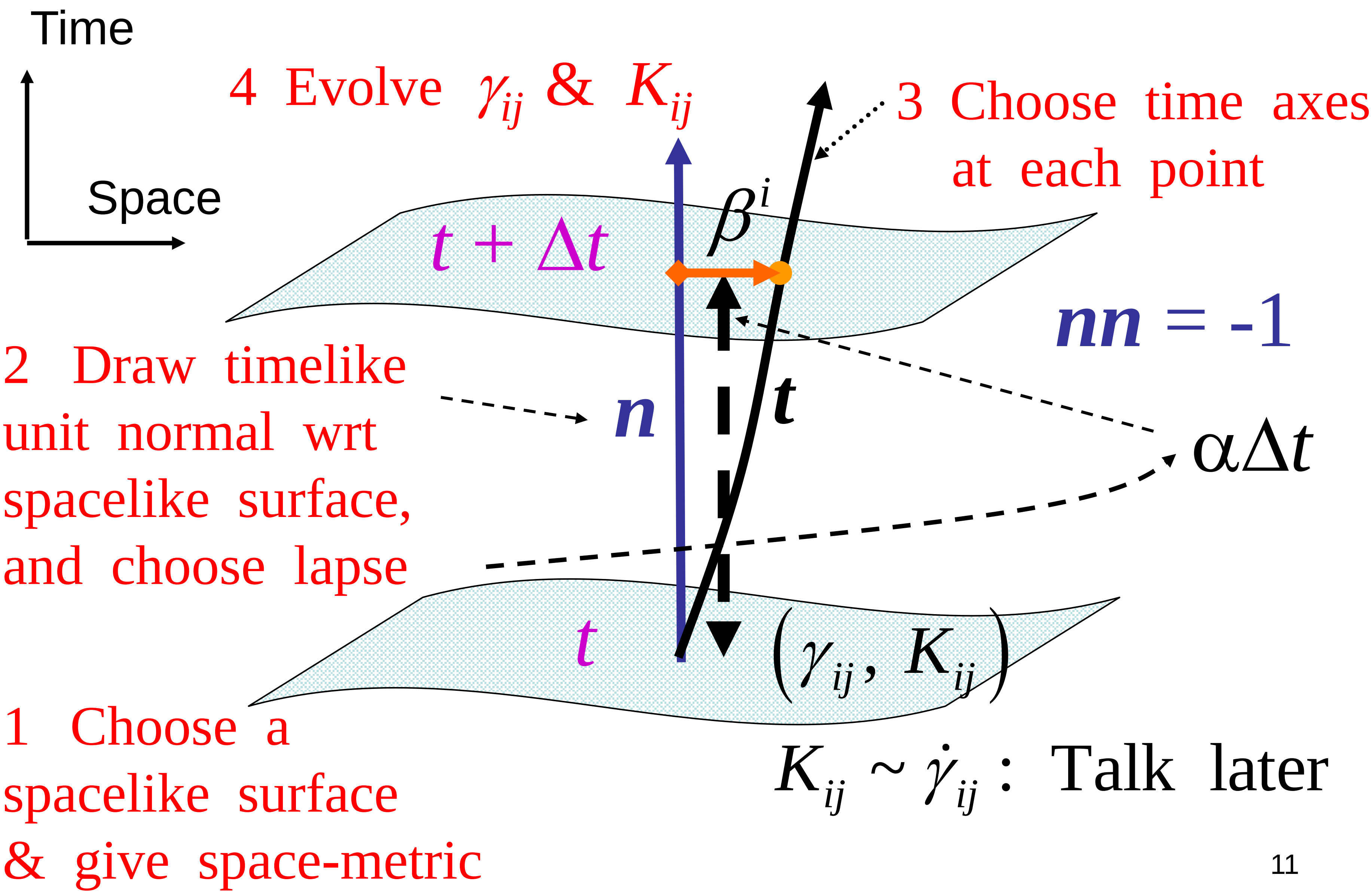
Section I: 3+1 (ADM) formalism

<u>Concept</u>

1. Foliate spacetime by spacelike surfaces
2. "Choose" time coordinates for a direction of time at each location
3. Follow dynamics of spacelike surfaces



3+1 decomposition (N+1)



Definition of variables

$g_{\mu\nu} \rightarrow (\gamma_{ij}, \alpha, \beta^k)$

γ_{ij} = space metric
K_{ij} = extrinsic curvature

Dynamics

α = lapse function
β^i = shift vector

Gauge

Lie derivative

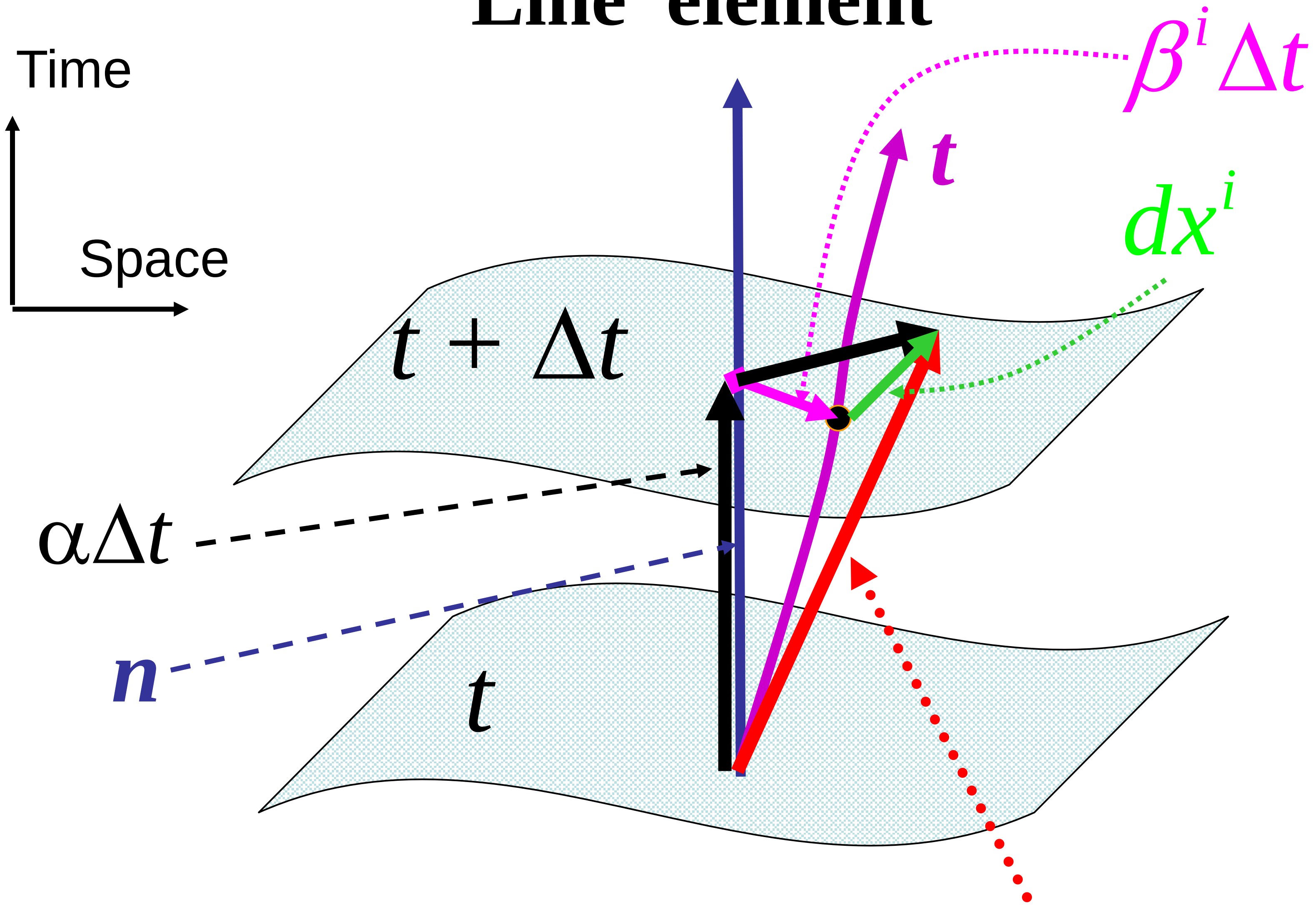
$$\gamma_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu : \quad \gamma_{i\mu} n^\mu = 0$$

$$K_{ij} \equiv -\gamma_i^\mu \gamma_j^\nu \nabla_\mu n_\nu = -\frac{1}{2} L_n \gamma_{ij}$$

$$\left[= -\frac{1}{2\alpha} \left(\partial_t \gamma_{ij} - D_i \beta_j - D_j \beta_i \right) \right]$$

$$\alpha n^\mu = t^\mu - \beta^\mu ; \quad n_\mu \beta^\mu = 0$$

Line element



$$ds^2 = -\left(\alpha dt\right)^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

Structure of variables

$$n^{\mu} = \left(\frac{1}{\alpha}, \ -\frac{\beta^i}{\alpha} \right), \quad n_{\mu} = \left(-\alpha, \ 0 \right); \ \text{ cf. } \ t^{\mu} = \left(1, \ 0 \right)$$

α : lapse function, β^i : shift vector; $\beta_i = \gamma_{ij} \beta^j$

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k, \beta_i \\ * \qquad\qquad\qquad , \gamma_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2, \beta^i/\alpha^2 \\ * \qquad\qquad , \gamma^{ij} - \beta^i \beta^j / \alpha^2 \end{pmatrix}$$

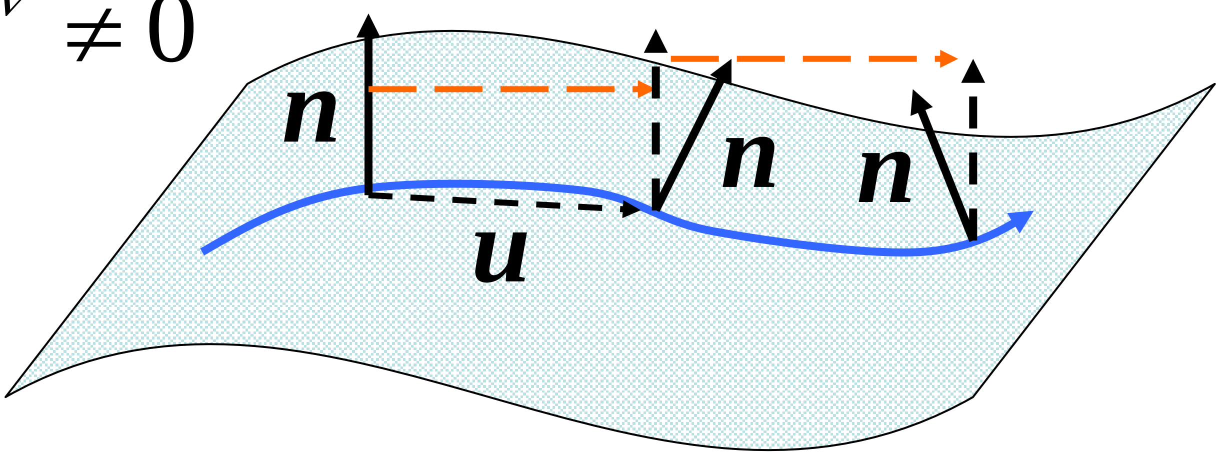
$$\text{Cf.} \quad \gamma_{\mu\nu} = \begin{pmatrix} \beta^k \beta_k, \beta_i \\ * \qquad , \gamma_{ij} \end{pmatrix}, \quad \gamma^{\mu\nu} = \begin{pmatrix} 0 \ , 0 \\ * \ , \gamma^{ij} \end{pmatrix}$$

$$K_{\mu\nu} = \begin{pmatrix} K_{ij} \beta^j \beta^j, K_{ij} \beta^j \\ * \qquad\qquad , K_{ij} \end{pmatrix}, \quad K^{\mu\nu} = \begin{pmatrix} 0 \ , 0 \\ * \ , K^{ij} \end{pmatrix}$$

Geometric meaning of K_{ij}

If space-like hyper-surface is curved,
n is not parallel-transported.

$$u^\mu \nabla_\mu n^\nu \neq 0$$



$$u^i K_{ij} = -\gamma^i_j u^\mu \nabla_\mu n_i \neq 0$$

K_{ij} denotes the "curved" degree
of chosen spatial hyper-surfaces

Next step is to rewrite Einstein's equation by γ_{ij} & K_{ij}

→ As a first step, it is necessary to define covariant derivative associated with γ_{ij}

$D_i \gamma_{jk} = 0 \quad \cdots$ Required property

$T^{ijk\cdots}{}_{lmn\cdots}$: A spacetime tensor

Define $D_h T^{ijk\cdots}{}_{lmn\cdots} = \underbrace{\gamma_h^{h'} \gamma_{i'}^{i} \gamma_{j'}^{j} \gamma_{k'}^{k}}_{\text{projection}} \cdots \underbrace{\gamma_l^{l'} \gamma_m^{m'} \gamma_n^{n'}}_{\text{projection}} \nabla_{h'} T^{i'j'k'\cdots}{}_{l'm'n'\cdots}$

Then,

$$D_i \gamma_{jk} = \gamma_i^{i'} \gamma_j^{j'} \gamma_k^{k'} \nabla_{i'} \gamma_{j'k'} = \gamma_i^{i'} \gamma_j^{j'} \gamma_k^{k'} \nabla_{i'} (g_{j'k'} + n_{j'} n_{k'})$$

$$= \gamma_i^{i'} \gamma_j^{j'} \gamma_k^{k'} \left[n_{j'} \nabla_{i'} n_{k'} + n_{k'} \nabla_{i'} n_{j'} \right] = 0 \qquad \text{OK}$$

3+1 definition of K_{ij}

$$K_{ij} = -\gamma_i^\alpha \gamma_j^\beta \nabla_\alpha n_\beta = -\frac{1}{2}\gamma_i^\alpha \gamma_j^\beta \left(\nabla_\alpha n_\beta + \nabla_\beta n_\alpha\right)$$

$$\gamma_i^\alpha \gamma_j^\beta \nabla_\alpha n_\beta = \left(g_i^\alpha + n_i n^\alpha\right) g_j^\beta \nabla_\alpha n_\beta \qquad \left(n^\beta \nabla_\alpha n_\beta = 0\right)$$

$$= \nabla_i n_j + n_i n^\mu \nabla_\mu n_j$$

$$\Rightarrow K_{ij} = -\frac{1}{2}\left(\nabla_i n_j + \nabla_j n_i + n^\mu \nabla_\mu \left(n_i n_j\right)\right)$$

$$= -\frac{1}{2}\left(\gamma_{j\alpha} \nabla_i n^\alpha + \gamma_{i\alpha} \nabla_j n^\alpha + n^\mu \nabla_\mu \gamma_{ij}\right)$$

$$= -\frac{1}{2}\left(\gamma_{j\alpha} \partial_i n^\alpha + \gamma_{i\alpha} \partial_j n^\alpha + n^\mu \partial_\mu \gamma_{ij}\right)$$

$$= -\frac{1}{2\alpha}\left(\partial_t \gamma_{ij} - D_i \beta_j - D_j \beta_i\right)$$

Usuful, often used relations

$$K_{\mu\nu} \equiv -\nabla_\mu n_\nu - n_\mu n^\sigma \nabla_\sigma n_\nu = -\nabla_\mu n_\nu - n_\mu D_\nu \ln\alpha$$

$Note:\ \ n^\sigma \nabla_\sigma n_\nu = D_\nu \ln\alpha:$ acceleration

We will use it for many times

3+1 decomposition of Einstein's equation

$$\begin{cases} G_{\mu\nu} n^\mu n^\nu = 8\pi T_{\mu\nu} n^\mu n^\nu & : \text{Hamiltonian constraint} \\ G_{\mu\nu} n^\mu \gamma^\nu_k = 8\pi T_{\mu\nu} n^\mu \gamma^\nu_k & : \text{Momentum constraint} \end{cases}$$

$$\left\{ G_{\mu\nu} \gamma^\mu_i \gamma^\nu_j = 8\pi T_{\mu\nu} \gamma^\mu_i \gamma^\nu_j : \text{Evolution equation} \right.$$

- First two eqs = constraint eqs
  ∵ no second derivative of spatial metric
- Last one = evolution eqs
  ∵ hyperbolic eqs of spatial metric
- No time derivative for α & β^k

Similar to Maxwell's eqs

$$\left.\begin{aligned} \nabla_i E^i &= 4\pi\rho_e \\ \nabla_i B^i &= 0 \end{aligned}\right\} \text{Constraint eqs}$$

$$\left.\begin{aligned} \partial_t E^i &= (\nabla \times B)^i - 4\pi j^i \\ \partial_t B^i &= -(\nabla \times E)^i \end{aligned}\right\} \text{Evolution eqs}$$

Step 1: Give an initial condition which satisfies constraints.

Step 2: Solve evolution equations.

All eqs have to be written by (γ, K)

- Method: Derive Gauss-Codacci equations

Start from definition of 3D Riemann tensor

$$\left(D_i D_j - D_j D_i\right)\omega_k = {}^{(3)}R_{ijk}{}^{l}\omega_l \; : \; \omega_l = 3D \text{ vector}$$

Using defintion of 3D covariant derivative,

$$D_i D_j \omega_k = \gamma_i^a \gamma_j^b \gamma_k^c \nabla_a \left(\gamma_b^l \gamma_c^m \nabla_l \omega_m\right)$$

$$= \gamma_i^a \gamma_j^b \gamma_k^c \nabla_a \nabla_b \omega_c + \gamma_i^a \gamma_j^b \gamma_k^m \left(\nabla_a \gamma_b^l\right)\left(\nabla_l \omega_m\right)$$

$$+ \gamma_i^a \gamma_j^l \gamma_k^c \left(\nabla_a \gamma_c^m\right)\left(\nabla_l \omega_m\right)$$

$$= \gamma_i^a \gamma_j^b \gamma_k^c \nabla_a \nabla_b \omega_c - \left(n^a \nabla_a \omega_m\right)\gamma_k^m K_{ij} - K_{ik} K_j^c \omega_c$$

where we used

$$\nabla_c \gamma_a^b = \nabla_c \left(g_a^b + n_a n^b \right) = \nabla_c \left(n_a n^b \right) = n_a \nabla_c n^b + n^b \nabla_c n_a$$

$$= -n_a \left(K_c^b + n_c D^b \ln \alpha \right) - n^b \left(K_{ac} + n_c D_a \ln \alpha \right)$$

$$\Rightarrow \quad \gamma_i^c \gamma_j^a \nabla_c \gamma_a^b = -n^b K_{ij}$$

$$\gamma_j^l n^m \nabla_l \omega_m = -\gamma_j^l \omega_m \nabla_l n^m = \omega_m K_j^m \qquad \left(n^a \omega_a = 0 \right)$$

Then, $\left(D_i D_j - D_j D_i \right) \omega_k$ gives

$${}^{(3)}R_{ijk}{}^l \omega_l = \gamma_i^a \gamma_j^b \gamma_k^c R_{abc}{}^l \omega_l - \omega_l K_j^l K_{ik} + \omega_l K_i^l K_{jk}$$

$$\Rightarrow \quad {}^{(3)}R_{ijkl} = \gamma_i^a \gamma_j^b \gamma_k^c \gamma_l^d R_{abcd} + K_{il} K_{jk} - K_{ik} K_{jl}$$

Gauss Codacci eq.

Note: ${}^{(N)}R_{ijkl} = \gamma_i^a \gamma_j^b \gamma_k^c \gamma_l^d \, {}^{(N+1)}R_{abcd} + K_{il} K_{jk} - K_{ik} K_{jl}$

Gauss-Codacci equations I, continued

$$^{(3)}R_{ijkl} = \gamma_i^a \gamma_j^b \gamma_k^c \gamma_l^d R_{abcd} + K_{il}K_{jk} - K_{ik}K_{jl}$$

Multiplying γ^{ik}

$$^{(3)}R_{jl} = \gamma_j^b \gamma_l^d \left(R_{bd} + R_{abcd} n^a n^c \right) + K_{jk}K_l^k - KK_{jl}$$

This will be used later

Multiplying γ^{jl} again

$$^{(3)}R = \left(R + 2R_{ad} n^a n^c \right) + K_{il}K^{il} - K^2$$

Here, $R + 2R_{ad} n^a n^c = 2G_{ab} n^a n^c = 16\pi T_{ab} n^a n^c$

$$\Rightarrow \quad ^{(3)}R + K^2 - K_{il}K^{il} = 16\pi T_{ab} n^a n^c$$

$\cdots$ Hamiltonian constraint

1 component

- **Derive Gauss-Codacci equations II**

Start from

$$D_i D_j n^k = \gamma_i^a \gamma_j^b \gamma_c^k \nabla_a D_b n^c = \gamma_i^a \gamma_j^b \gamma_c^k \nabla_a \left(\gamma_b^d \gamma_e^c \nabla_d n^e \right)$$

$$= \gamma_i^a \gamma_j^b \gamma_c^k \nabla_a \nabla_b n^c + \gamma_i^a \gamma_j^b \gamma_c^k \left(\nabla_a \gamma_b^d \right) \nabla_d n^c$$

$$+ \gamma_i^a \gamma_j^b \gamma_c^k \left(\nabla_a \gamma_e^c \right) \nabla_b n^e$$

$$= \gamma_i^a \gamma_j^b \gamma_c^k \nabla_a \nabla_b n^c - \gamma_c^k K_{ij} n^d \nabla_d n^c - \gamma_j^b K_i^k \underline{n_e \nabla_b n^e}$$

$$= \gamma_i^a \gamma_j^b \gamma_c^k \nabla_a \nabla_b n^c - \left(D^k \ln \alpha \right) K_{ij}$$

where $n^d \nabla_d n^c = D^c \ln \alpha, \quad n_e \nabla_b n^e = 0$

• Derive Gauss-Codacci equations II

$$D_i D_j n^k = \gamma_i^a \gamma_j^b \gamma_c^k \nabla_a \nabla_b n^c - \left(D^k \ln\alpha\right) K_{ij}$$

$$\begin{matrix}(i,k) \\ \Rightarrow \\ (j,k)\end{matrix} \quad \begin{cases} D_i D_j n^i = \gamma_c^a \gamma_j^b \nabla_a \nabla_b n^c - \left(D^i \ln\alpha\right) K_{ij} \\ D_j D_i n^i = \gamma_j^a \gamma_c^b \nabla_a \nabla_b n^c - \left(D^i \ln\alpha\right) K_{ij} \end{cases}$$

$$\Rightarrow \quad D_i D_j n^i - D_j D_i n^i = \gamma_c^a \gamma_j^b \left(\nabla_a \nabla_b - \nabla_b \nabla_a\right) n^c$$

$$= -\gamma_c^a \gamma_j^b R_{abd}{}^{\mathrm{c}} n^d$$

$$D_i n^i = -K, \quad D_j n^i = -K^i_j, \quad \gamma_c^a \gamma_j^b R_{abd}{}^{\mathrm{c}} n^d = -R_{bd} \gamma_j^b n^d$$

$$\Rightarrow \quad D_i K^i_j - D_j K = -R_{bd} \gamma_j^b n^d = -8\pi T_{bd} \gamma_j^b n^d$$

$\cdots$ Momentum constraint

3 components

Derive evolution equation

Start from contracted Gauss-Codacci eq. I

$$ {}^{(3)}R_{jl} = \gamma^b_j \gamma^d_l \left(R_{bd} + R_{abcd} n^a n^c \right) + K_{il} K^i_j - K K_{jl} $$

$$ \Rightarrow \ R_{abcd} n^a n^c \ \text{contains} \ \partial_t^2 \gamma_{ij} \ \text{or} \ \partial_t K_{ij} $$

Let us calculate $R_{abcd} n^d = \left(\nabla_a \nabla_b - \nabla_b \nabla_a \right) n_c$

$$ \nabla_a \nabla_b n_c = \nabla_a \left(-K_{bc} - n_b D_c \ln \alpha \right) $$
$$ = -\nabla_a K_{bc} + \left(K_{ab} + n_a D_b \ln \alpha \right) D_c \ln \alpha $$
$$ - n_b \nabla_a D_c \ln \alpha $$

Remember: $\nabla_b n_c = -K_{bc} - n_b D_c \ln \alpha$

From $\left(\nabla_a \nabla_b - \nabla_b \nabla_a\right) n_c = R_{abc}{}^{d} n_d \quad \Rightarrow$

$$\gamma_j^a \gamma_l^c R_{abcd} n^b n^d = \gamma_j^a \gamma_l^c n^b \left(-\nabla_a K_{bc} + \nabla_b K_{ac}\right)$$

$$+\left(D_j \ln \alpha\right) D_l \ln \alpha + D_j D_l \ln \alpha$$

$$-\gamma_j^a \gamma_l^c n^b \nabla_a K_{bc} = \gamma_j^a K_{bl} \nabla_a n^b$$

$$= -\gamma_j^a K_{bl} \left(K_a^b + n_a D^b \ln \alpha\right) = -K_{kl} K_j^k$$

$$\gamma_j^a \gamma_l^c n^b \nabla_b K_{ac} = \gamma_j^a \gamma_l^c \left(L_n K_{ac} - K_{ab} \nabla_c n^b - K_{bc} \nabla_a n^b\right)$$

$$= L_n K_{jl} + 2 K_{jk} K_l^k$$

$$\therefore \quad \underline{\gamma_j^a \gamma_l^c R_{abcd} n^b n^d = L_n K_{jl} + K_{jk} K_l^k + \alpha^{-1} D_j D_l \alpha}$$

Thus, the G-C equation becomes

$$^{(3)}R_{jl} = \gamma_j^b \gamma_l^d R_{bd} + L_n K_{jl} + \alpha^{-1} D_j D_l \alpha + 2K_{kl} K_j^k - K K_{jl}$$

Here,

$$L_n K_{ij} = \alpha^{-1}\left[\partial_t K_{ij} - \beta^k D_k K_{ij} - K_{ik} D_j \beta^k - K_{jk} D_i \beta^k\right]$$

$$\therefore \quad \partial_t K_{ij} = \alpha\, {}^{(3)}R_{ij} - D_i D_j \alpha + \alpha\left(K K_{ij} - 2K_{ik} K_j^k\right) + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{jk} D_i \beta^k - \alpha \gamma_i^b \gamma_j^d R_{bd}$$

Note: $\gamma_i^b \gamma_j^d R_{bd} = 8\pi\left(\gamma_i^b \gamma_j^d T_{bd} - \frac{\gamma_{ij}}{2} T\right)$

Evolution equations = 6 components

Summary of 3+ 1 formalism

$$ {}^{(3)}R - K_{ij}K^{ij} + K^2 = 16\pi T_{\mu\nu} n^\mu n^\nu $$

$$ D_i K^i_j - D_j K = -8\pi T_{\mu\nu} n^\mu \gamma^\nu_j $$

Constraints

$$ \dot{K}_{ij} = \alpha\left({}^{(3)}R_{ij} + KK_{ij} - 2K_{il}K^l_j \right) - D_i D_j \alpha $$

$$ + K_{il} D_j \beta^l + K_{jl} D_i \beta^l + \beta^l D_l K_{ij} $$

Evolution

$$ -8\pi\alpha T_{\mu\nu}\left[\gamma^\mu_i \gamma^\nu_j - \frac{1}{2}(\gamma^{\mu\nu} - n^\mu n^\nu)\gamma_{ij} \right] $$

$$ \dot{\gamma}_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i $$

$\alpha, \beta^i \;\Leftarrow\;$ Gauge condition

Constrained system

- ADM equations seem to have too many components: Constraints seem to be redundant equations, because γ_{ij} & K_{ij} are determined by solving evolution equations

- Constraints are guaranteed to be satisfied if evolution equations are solved correctly
  → **No inconsistency**

Evolution of constraints

$$A_{\mu\nu} \equiv G_{\mu\nu} - 8\pi T_{\mu\nu}$$

$$= H_0 n_\mu n_\nu + H_i \gamma^i_\mu n_\nu + H_i \gamma^i_\nu n_\mu + H_{ij} \gamma^i_\mu \gamma^i_\nu$$

$H_0 = 0,\ \ H_i = 0 \ \cdots$ H & M Constraints

$H_{ij} = 0 \qquad \cdots$ Evolution eqs.

$$\nabla_\mu A^\mu_\nu = 0 \quad \Rightarrow$$

$$\left(\partial_t - \beta^l \partial_l\right) H_0 = \alpha K H_0 - 2 H_i D^i \alpha - \alpha D_i H^i + \alpha H_{ij} K^{ij}$$

$$\left(\partial_t - \beta^l \partial_l\right) H_i = -H_0 D_i \alpha + \alpha K H_i + H_k \beta^k{}_{,i} - D_k\left(\alpha H^k_i\right)$$

If constrains are zero at $t = 0$ and evolution equations are satisfied for any t,

constraints are always satisfied.

Nature of standard 3+1 formalism

- Evolution equations are wave equations of 6 components, but *it is not simple one*: Many additional terms even in linear order

$$\dot{K}_{ij} \sim -\frac{1}{2}\ddot{\gamma}_{ij}$$

$$^{(3)}R_{ij} \sim \frac{1}{2}\left[-\Delta\gamma_{ij} + \overbrace{\gamma^{kl}\left(\gamma_{ik,lj} + \gamma_{jk,li} - \partial_{ij}\gamma_{kl}\right)}\right]$$

$$\Rightarrow \quad \ddot{\gamma}_{ij} \approx \Delta\gamma_{ij} + \cdots$$

cf. Maxwell's equation

$$\partial_\alpha \partial^\alpha A_\beta - \underline{\partial_\beta \partial_\alpha A^\alpha} = 0$$

Linearized equations

Linearized Einstein equations

with $\alpha = 1$ & $\beta^i = 0$

$$\gamma_{ij} = \delta_{ij} + h_{ij}; \quad |h_{ij}| \ll 1$$

Evolution eq. : $\ddot{h}_{ij} = \Delta h_{ij} - \underline{h_{ik,kj} - h_{jk,ki} + h_{kk,ij}}$

Constraint H : $\Delta h_{ii} - h_{ik,ki} = 0$

M : $\dot{h}_{ij,i} - \dot{h}_{ii,j} = 0$

This causes a problem

Stability analysis

Decomposition:

$$h_{ij} = A\delta_{ij} + C_{,ij} + 2B_{(i,j)} + h_{ij}^{\mathrm{TT}}$$

$A,\ C$: scalar, B_i : vector, h_{ij}^{TT} : tensor

definition: $B_{i,i} = 0, \quad h_{ij,j}^{\mathrm{TT}} = h_{ii}^{\mathrm{TT}} = 0$

$\Rightarrow$ Trace $h_{ii} = 3A + \Delta C,$

Divergence $h_{ij,j} = A_{,i} + \Delta(C_{,i} + B_i)$

$$\xrightarrow{\text{substitute}} \quad \ddot{h}_{ij} = \Delta h_{ij} - h_{ik,kj} - h_{jk,ki} + h_{kk,ij}$$

$$\Delta h_{ii} - h_{ik,ki} = 0, \quad \dot{h}_{ij,i} - \dot{h}_{ii,j} = 0$$

3+1 Equations

$$\text{Constraints}: \begin{cases} \text{H:} \quad \Delta A = 0 \\ \text{M:} \ \partial_t \left(-2A_{,i} + \Delta B_i \right) = 0 \end{cases}$$

$$\text{Evolution eqs.}: \begin{cases} \ddot{h}_{ij}^{\mathrm{TT}} = \Delta h_{ij}^{\mathrm{TT}} \\ \ddot{B}_{(i,j)} = 0 \\ \ddot{A} = \Delta A \\ \ddot{C} = A \end{cases}$$

Wave equation

Strange forms

Solutions I

1 Equations for h_{ij}^{TT} =Wave equations

$\Rightarrow$ True degree of GWs: No problem

2 Constraint (H) : $A = 0$

& Evolution equation for $A =$ wave eq.

$\Rightarrow$ Violated constraint will propagate away.

3 Constraint (M) : $\left(-2A_{,i} + \Delta B_i\right) = F\left(x^i\right)$

For $A = 0$, $B_i = F_{B_i}\left(x^i\right)$

Numerical integration of zero is zero

Evolution equation for B_i gives $\dot{B}_i = F_B\left(x^i\right) \to 0$

& $B_i = \left\{F_B\left(x^i\right)t\right\} + F_{B_i}\left(x^i\right)$

Perhaps no problem

Solutions II

C is not constrainted by constraints,

but determined by $\ddot{C} = A, \ \ddot{A} = \Delta A$

If constraint is violated and $A \neq 0$ initially,

$$A = \sum_{l,m} Y_{lm} \frac{\ddot{f}_{lm}(r-t) + \ddot{g}(r+t)}{r}$$

$$\Rightarrow C = \sum Y_{lm} \frac{f(r-t) + g(r+t)}{r} + C_1 t + C_2$$

Constraint violation is serious in this case

Small error in *A* results in serious error in *C*

To summarize

- In the original 3+1 (N+1) formalism, *if constraints are violated even slightly*, the error increases with time *even in a nearly flat spacetime* with no limit
- Namely, **it is unsuitable for numerical relativity**
- **Source:**

$$\ddot{h}_{ij} = \Delta h_{ij} \underbrace{- h_{ik,kj} - h_{jk,ki} + h_{kk,ij}}$$

First, realized by T. Nakamura (1987)

Section II: BSSN formalism

Essence

- **Need reformulate of 3+1 formalism**
- At least, in the linear level, constraint violation mode must not appear

Define new variables

$$\begin{cases} F_i = h_{ij,j} \\ \Phi = h_{ii} \end{cases}$$

Evolution eqs. for F_i and Φ ?

and rewrite as

$$\ddot{h}_{ij} = \Delta h_{ij} - F_{i,j} - F_{j,i} + \Phi_{,ij}$$

Reformulation using constraint equations

Momentum constraint: $\dot{h}_{ij,j} - \dot{h}_{jj,i} = 0$

$\Rightarrow \quad \dot{F}_i - \dot{\Phi}_{,i} = 0$: Evolution eq for F_i

Trace of $\ddot{h}_{ij} = \Delta h_{ij} - F_{i,j} - F_{j,i} + \Phi_{,ij}$

$\Rightarrow \quad \ddot{\Phi} = 2\Delta\Phi - 2F_{i,i}$

Hamiltonian constraint: $\Delta\Phi - F_{i,i} = 0$

$\Rightarrow \quad \ddot{\Phi} = 0 \quad \Rightarrow \quad \dot{\Phi} = 0 \Rightarrow \dot{F}_i = 0$

$\Rightarrow \quad \ddot{h}_{ij} = \Delta h_{ij}$

No problem !!

Reformulation increasing the variables and using constraints appears to be robust

- Similar definition of new variables F_i & Φ is possible in the non-linear case

First, derived by T. Nakamura (1987)
Subsequently modified by Shibata (1995), Baumgarate and Shapiro (1998)

Original BSSN formalism

(Shibata-Nakamura 1995)

First of all, write the line element

$$ds^2 = -\left(\alpha^2 - \beta_i \beta^i\right) dt^2 + 2\beta_i dx^i dt + e^{4\phi} \tilde{\gamma}_{ij} dx^i dx^j$$

Here, $\det\left(\tilde{\gamma}_{ij}\right) = 1.$ (ϕ corresponds to Φ.)

As conjugates for $\tilde{\gamma}_{ij}$ and ϕ, define

$$\tilde{A}_{ij} = e^{-4\phi}\left(K_{ij} - \frac{1}{3}\gamma_{ij} K\right) \text{ and } K = \text{trace}\left(K_{ij}\right) = \gamma^{ij} K_{ij}$$

Up to here, we increase 2 variables (ϕ, K) and two constraints, $\det\left(\tilde{\gamma}_{ij}\right) = 1$ and $\tilde{A}_{ij}\tilde{\gamma}_{ij} = 0$

Then, the equations are

$$(\partial_t - \beta^l \partial_l)\tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{il}\beta^l{}_{,j} + \tilde{\gamma}_{jl}\beta^l{}_{,i} - \frac{2}{3}\tilde{\gamma}_{ij}\beta^l{}_{,l}$$

$$(\partial_t - \beta^l \partial_l)\phi = \frac{1}{6}\left(-\alpha K + \beta^l{}_{,l}\right)$$

$$(\partial_t - \beta^l \partial_l)\tilde{A}_{ij} = \alpha e^{-4\phi}\left(R_{ij} - \frac{1}{3}\gamma_{ij}R\right) - e^{-4\phi}\left(D_i D_j \alpha - \frac{1}{3}\gamma_{ij}\Delta\alpha\right)$$
$$+\alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j\right) + \tilde{A}_{il}\partial_j\beta^l + \tilde{A}_{jl}\partial_i\beta^l - \frac{2}{3}\beta^l{}_{,l}\tilde{A}_{ij}$$
$$-8\pi\alpha e^{-4\phi}T_{\mu\nu}\left[\gamma_i^\mu\gamma_j^\nu - \frac{1}{3}\gamma^{\mu\nu}\gamma_{ij}\right]$$

Used H-constraint

$$(\partial_t - \beta^l \partial_l)K = \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) - \Delta\alpha + 4\pi\alpha T_{\mu\nu}\left(n^\mu n^\nu + \gamma^{\mu\nu}\right)$$

Not sufficient in this stage !

Note

- The linear analysis for the simple conformally-decomposed formalism shows "**System is even more unstable**"
- **An exponentially growing mode appear**

$$\ddot{h}_{ij} = \Delta h_{ij} - h_{ik,kj} - h_{jk,ki}$$

$$h_{ij} = C_{,ij} + B_{(i,j)} + h_{ij}^{\mathrm{TT}}$$

$$\Rightarrow \quad \ddot{C} = -\Delta C$$

Linear analysis shows that the problems come from R_{ij}

$R_{ij} = \tilde{R}_{ij} + R^{\phi}_{ij}$: $\tilde{R}_{ij}$ is Ricci tensor of $\tilde{\gamma}_{ij}$

$$R^{\phi}_{ij} = -2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{\gamma}_{ij}\tilde{D}_k\tilde{D}^k\phi + 4\left(\tilde{D}_i\phi\right)\tilde{D}_j\phi$$
$$-4\tilde{\gamma}_{ij}\left(\tilde{D}_k\phi\right)\tilde{D}^k\phi \quad \cdots \text{ scalar part}$$

$$\tilde{R}_{ij} = \frac{1}{2}\left[-\tilde{\gamma}^{kl}\left(\tilde{\gamma}_{ij,kl} - \tilde{\gamma}_{ik,jl} - \tilde{\gamma}_{jk,il}\right)\right]$$
$$+\tilde{\gamma}^{kl}{}_{,k}\tilde{\Gamma}_{l,ij} - \tilde{\Gamma}^{l}{}_{jk}\tilde{\Gamma}^{k}{}_{il}$$

Already rewritten by ϕ

See appendix

Write formally $\tilde{\gamma}^{ij} = \delta^{ij} + f^{ij}$

$$\Rightarrow \tilde{\gamma}^{kl}\left(\tilde{\gamma}_{ij,kl} - \tilde{\gamma}_{ik,jl} - \tilde{\gamma}_{jk,il}\right)$$

$$= \Delta_{\text{flat}}\tilde{\gamma}_{ij} - \delta^{kl}\left(\tilde{\gamma}_{ik,jl} + \tilde{\gamma}_{jk,il}\right) \quad \leftarrow \text{Linear}$$

$$+ f^{kl}\left(\tilde{\gamma}_{ij,kl} - \tilde{\gamma}_{ik,jl} - \tilde{\gamma}_{jk,il}\right) \quad \leftarrow \text{nonlinear}$$

$\Rightarrow$ Define $F_i = \delta^{kl}\tilde{\gamma}_{ik,l}$

as in the linear case

$$\Rightarrow \Delta_{\text{flat}}\tilde{\gamma}_{ij} - \delta^{kl}\left(\tilde{\gamma}_{ik,jl} + \tilde{\gamma}_{jk,il}\right)$$

$$= \Delta_{\text{flat}}\tilde{\gamma}_{ij} - \left(F_{i,j} + F_{j,i}\right)$$

Next step: Derive equations for the new variables *using constraints*

- As in the linear case, the equation for F_i should be derived from momentum constraint

Momentum constraint:

$$\tilde{D}_i\left(e^{6\phi}\tilde{A}^i_j\right)-\frac{2}{3}e^{6\phi}\tilde{D}_jK=8\pi J_je^{6\phi}$$

or $$\tilde{D}_i\left(\alpha\tilde{A}^i_j\right)+\left\{6\alpha\left(\tilde{D}_i\phi\right)-\tilde{D}_i\alpha\right\}\tilde{A}^i_j-\frac{2}{3}\alpha\tilde{D}_jK=8\pi\alpha J_j$$

Here, $$2\alpha\tilde{A}_{ij}=-(\partial_t-\beta^l\partial_l)\tilde{\gamma}_{ij}+\tilde{\gamma}_{il}\beta^l{}_{,j}+\tilde{\gamma}_{jl}\beta^l{}_{,i}-\frac{2}{3}\tilde{\gamma}_{ij}\beta^l{}_{,l}$$

Thus, a term $\tilde{D}_i\left(\tilde{\gamma}^{ik}\partial_t\tilde{\gamma}_{jk}\right)$ appears and

$$\tilde{D}_i\left(\tilde{\gamma}^{ik}\partial_t\tilde{\gamma}_{jk}\right)=\tilde{\gamma}^{ik}\left(\partial_i\partial_t\tilde{\gamma}_{jk}-\tilde{\Gamma}^l_{ij}\partial_t\tilde{\gamma}_{kl}-\tilde{\Gamma}^l_{ik}\partial_t\tilde{\gamma}_{jl}\right)$$

$$=\partial_t F_j+f^{ik}\partial_i\partial_t\tilde{\gamma}_{jk}-\tilde{\gamma}^{ik}\left(\tilde{\Gamma}^l_{ij}\partial_t\tilde{\gamma}_{kl}+\tilde{\Gamma}^l_{ik}\partial_t\tilde{\gamma}_{jl}\right)$$

Namely, momentum constraint can be regarded as the evolution equation for F_i

Other terms with $\partial_t\tilde{\gamma}_{jk}$ are rewritten using

$$(\partial_t-\beta^l\partial_l)\tilde{\gamma}_{ij}=-2\alpha\tilde{A}_{ij}+\tilde{\gamma}_{il}\beta^l{}_{,j}+\tilde{\gamma}_{jl}\beta^l{}_{,i}-\frac{2}{3}\tilde{\gamma}_{ij}\beta^l{}_{,l}$$

Equation for F_i

$$
\begin{aligned}
&\left(\partial_t - \beta^k \partial_k\right) F_i \\
&= 2\alpha\left(f^{jk}\tilde{A}_{ij,k} + \tilde{\gamma}^{jk}{}_{,k}\tilde{A}_{ij} - \frac{1}{2}\tilde{A}^{jk}\tilde{\gamma}_{jk,i} + 6\phi_{,k}\tilde{A}_i^k - \frac{2}{3}K_{,i} \right) \\
&\quad - 2\delta^{jk}\alpha_{,k}\tilde{A}_{ij} + \delta^{jl}\beta^k{}_{,l}\tilde{\gamma}_{ij,k} \\
&\quad + \left(\tilde{\gamma}_{ik}\beta^k{}_{,j} + \tilde{\gamma}_{jk}\beta^k{}_{,i} - \frac{2}{3}\tilde{\gamma}_{ij}\beta^k{}_{,k} \right)_{,l} \delta^{jl} - 16\pi\alpha J_i
\end{aligned}
$$

Note no nonlinear term of $\left(h_{ij}, \tilde{A}_{ij}\right)$; $h_{ij} = \tilde{\gamma}_{ij} - \delta_{ij}$

Summary of BSSN formalism

- Definition of 3 additional variables (5 components) (F_i, K, ϕ) is essential.
- **Conformal transformation is *not* essential at all**: Only with conformal transformation, the resulting formalism does not work → "Conformal formalism" is misunderstanding (stupid) naming.
- The increase of variables results in the increase of new constraints: **Now, 17 equations with 9 constraint equations.**

Alternative (Baumgarte-Shapiro ,1998)

Define $\Gamma^i = -\tilde{\gamma}^{ij}{}_{,j}$ instead of $F_i = \delta^{jk}\tilde{\gamma}_{ij,k}$.

(In the linear level, both reduce to $h_{ij,j}$.)

$$\left(\partial_t - \beta^k\partial_k\right)\Gamma^i = 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} - \frac{2}{3}\tilde{\gamma}^{ij}K_{,j} + 6\phi_{,j}\tilde{A}^{ij}\right)$$

$$-\tilde{\Gamma}^j\beta^i_{,j} + \frac{2}{3}\tilde{\Gamma}^i\beta^j{}_{,j} + \tilde{\gamma}^{jk}\beta^i{}_{,jk} + \frac{1}{3}\tilde{\gamma}^{ik}\beta^j{}_{,jk}$$

$$-16\pi\alpha\tilde{\gamma}^{ij}J_j$$

Slightly simpler

$$\tilde{R}_{ij} = -\frac{1}{2}\left(\tilde{\gamma}^{kl}\tilde{\gamma}_{ij,kl} - \underline{\tilde{\gamma}_{ik}\Gamma^k{}_{,j} - \tilde{\gamma}_{jk}\Gamma^k{}_{,i}}\right)$$

$$-\frac{1}{2}\left(\tilde{\gamma}^{kl}{}_{,j}\tilde{\gamma}_{ik,l} + \tilde{\gamma}_{jk,l}\tilde{\gamma}^{kl}{}_{,i} - \tilde{\gamma}_{ij,k}\Gamma^k\right) - \tilde{\Gamma}^l_{ik}\tilde{\Gamma}^k_{jl}$$

Nine components of constraints

Hamiltonian constraint (1)

$$^{(3)}R - K_{ij}K^{ij} + K^2 = 16\pi T_{\mu\nu} n^{\mu} n^{\nu}$$

Momentum constraint (3)

$$D_i K^i_j - D_j K = -8\pi T_{\mu\nu} n^{\mu} \gamma^{\nu}_j$$

Tracefree condition for $\tilde{A}_{ij}$ (1)

$$\tilde{A}_{ij}\tilde{\gamma}^{ij} = 0$$

Determinant=1 for $\tilde{\gamma}_{ij}$ (1)

$$\det\left(\tilde{\gamma}_{ij}\right) = 1$$

Auxiliary variable (3)

$$F_i = \tilde{\gamma}_{ij,j} \quad \text{or} \quad \Gamma^i = -\tilde{\gamma}^{ij}{}_{,j}$$

Puncture-BSSN

(Campanelli et al. 2005)

Define $\chi = e^{-4\phi}$ (or $W = e^{-2\phi}$) instead of ϕ to follow a black hole spacetime.

Schwarzschild spacetime in the isotropic coordinates:

$$ds^2 = -\left(\frac{1 - M/2r}{1 + M/2r}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 \left(dx^2 + dy^2 + dz^2\right)$$

$$\phi = \ln\left(1 + \frac{M}{2r}\right) \xrightarrow{r \to 0} \infty$$

Define $\chi = \left(1 + \frac{M}{2r}\right)^{-4}$: regular everywhere

$\Rightarrow$ BH spacetime can be numerically followed with no special technique

New conformal factor

$$\chi = e^{-4\phi} \quad \text{or} \quad W = e^{-2\phi}$$

$$(\partial_t - \beta^l \partial_l)\phi = \frac{1}{6}\left(-\alpha K + \beta^l{}_{,l}\right)$$

$$\Rightarrow \quad (\partial_t - \beta^l \partial_l)W = \frac{W}{3}\left(\alpha K - \beta^l{}_{,l}\right)$$

No divergence

$$\psi^{-4} R_{ij}^{\psi}$$

$$= 2e^{-4\phi}\left[-\tilde{D}_i \tilde{D}_j \phi - \tilde{\gamma}_{ij}\tilde{\Delta}\phi + 2\tilde{D}_i\phi \tilde{D}_j\phi - 2\tilde{\gamma}_{ij}\tilde{D}_k\phi \tilde{D}^k\phi\right]$$

$$\Rightarrow \quad \psi^{-4} R_{ij}^{\psi} = W\tilde{D}_i \tilde{D}_j W + \tilde{\gamma}_{ij}\left(W\tilde{\Delta}W - 2\tilde{D}_k W \tilde{D}^k W\right)$$

No irregular term even for BH spacetime

Other prescriptions: Special gauge

Linearized Einstein equations

with $\alpha \neq 1$ & $\beta^i \neq 0$

$$\gamma_{ij} = \delta_{ij} + h_{ij}; \quad |h_{ij}| \ll 1, \qquad \alpha = 1 - a/2; \quad |a| \ll 1$$

Evolution eq. : $\ddot{h}_{ij} = \Delta h_{ij} - h_{ik,kj} - h_{jk,ki} + h_{kk,ij}$

$$- a_{,ij} + \dot{\beta}_{i,j} + \dot{\beta}_{j,i}$$

Constraint H : $\Delta h_{ii} - h_{ik,ki} = 0$

M : $\dot{h}_{ij,i} - \dot{h}_{ii,j} = 0$

Harmonic gauge:

$$\partial_\mu g^{\mu\nu} = 0 \Rightarrow \begin{cases} -\dot{a} + \beta^i{}_{,i} = 0 \\ \dot{\beta}_i - h_{ij,j} = 0 \end{cases}$$

$$-h_{ik,kj} - h_{jk,ki} + h_{kk,ij} - a_{,ij} + \dot{\beta}_{i,j} + \dot{\beta}_{j,i} = h_{kk,ij} - a_{,ij}$$

$$\therefore \quad \begin{cases} \ddot{h}_{ij} = \Delta h_{ij} + h_{kk,ij} - a_{,ij} \\ \ddot{a} = h_{ij,ji} \xrightarrow{\text{Constraint (H)}} \ddot{a} = \Delta h_{kk} \end{cases}$$

$$\Rightarrow \quad \partial_t^2 \left(h_{kk} - a \right) = \Delta \left(h_{kk} - a \right)$$

Only wave equations appear → No problem

This shows an evidence why Pretorius formulation works

Extension to N+1 case ($N > 3$)

- ADM equations are unchanged
- BSSN formalism is slightly modified because the dimension is different (Yoshino-Shibata 09)
- In the following, spacetime dimension is denoted by $D = N+1$

For $ds^2 = -\left(\alpha^2 - \beta_k \beta^k\right) dt^2 + 2\beta_k dx^k dt + \chi^{-1} \tilde{\gamma}_{ij} dx^i dx^j$

$$(\partial_t - \beta^l \partial_l)\tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{il}\beta^l{}_{,j} + \tilde{\gamma}_{jl}\beta^l{}_{,i} - \frac{2}{D-1}\tilde{\gamma}_{ij}\beta^l{}_{,l}$$

$$(\partial_t - \beta^l \partial_l)\chi = \frac{2\chi}{D-1}\left(\alpha K - \beta^l{}_{,l}\right)$$

$$(\partial_t - \beta^l \partial_l)\tilde{A}_{ij} = \alpha\chi\left(R_{ij} - \frac{1}{D-1}\gamma_{ij}R\right) - \chi\left(D_i D_j \alpha - \frac{1}{D-1}\gamma_{ij}\Delta\alpha\right)$$
$$+\alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j\right) + \tilde{A}_{il}\partial_j\beta^l + \tilde{A}_{jl}\partial_i\beta^l - \frac{2}{D-1}\beta^l{}_{,l}\tilde{A}_{ij}$$
$$-8\pi\alpha\chi T_{\mu\nu}\left[\gamma^\mu_i \gamma^\nu_j - \frac{1}{D-1}\gamma^{\mu\nu}\gamma_{ij}\right]$$

$$(\partial_t - \beta^l \partial_l)K = \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{K^2}{D-1}\right) - \Delta\alpha + \frac{8\pi\alpha}{D-2}T_{\mu\nu}\left(\left(D-3\right)n^\mu n^\nu + \gamma^{\mu\nu}\right)$$

$$\left(\partial_t - \beta^l \partial_l\right)\Gamma^i = 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} - \frac{D-2}{D-1}\tilde{\gamma}^{ij}K_{,j} - \frac{D-1}{2\chi}\chi_{,j}\tilde{A}^{ij}\right)$$

$$-\tilde{\Gamma}^j\beta^i_{,j} + \frac{2}{D-1}\tilde{\Gamma}^i\beta^j_{,j} + \tilde{\gamma}^{jk}\beta^i_{,jk} + \frac{D-3}{D-1}\tilde{\gamma}^{ik}\beta^j_{,jk}$$

$$-16\pi\alpha\tilde{\gamma}^{ij}J_j$$

$$R_{ij} = \tilde{R}_{ij} + R^{\chi}_{ij}$$

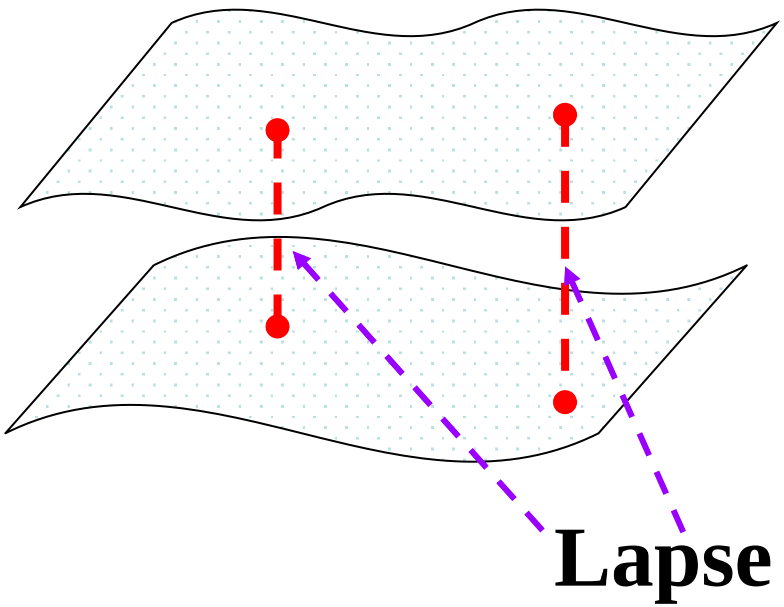
$$R^{\chi}_{ij} = \frac{D-3}{2\chi}\tilde{D}_i\tilde{D}_i\chi + \frac{1}{2\chi}\tilde{\gamma}_{ij}\tilde{D}_k\tilde{D}^k\chi - \frac{D-3}{4\chi^2}\left(\tilde{D}_i\chi\right)\tilde{D}_j\chi$$

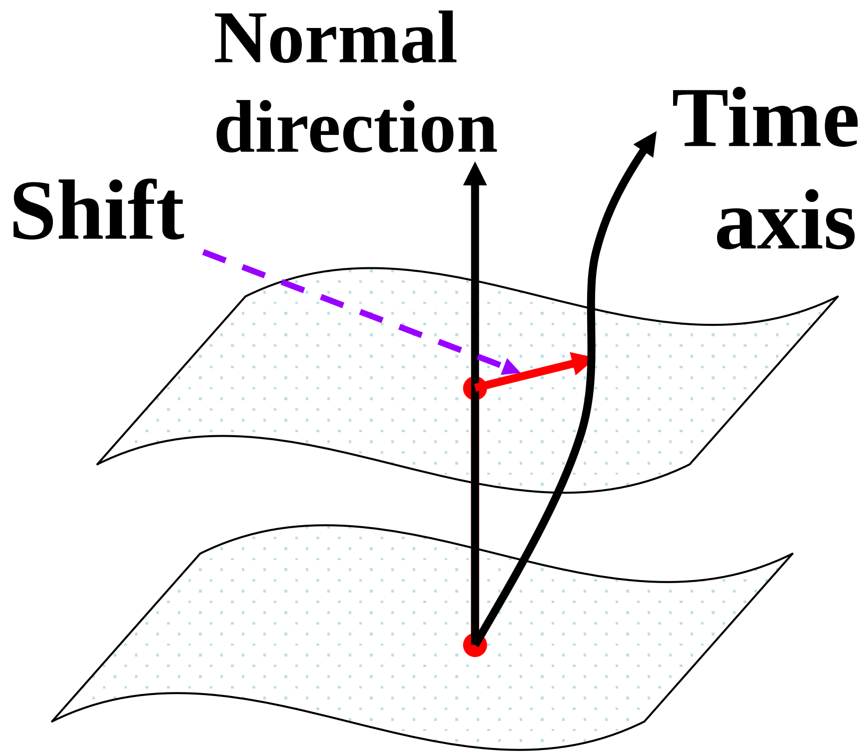
$$-\frac{D-1}{4\chi^2}\tilde{\gamma}_{ij}\left(\tilde{D}_k\chi\right)\tilde{D}^k\chi$$

Robust for any dimension (at least up to 7D, Shibata & Yoshino, '10)

Section III: Gauge conditions

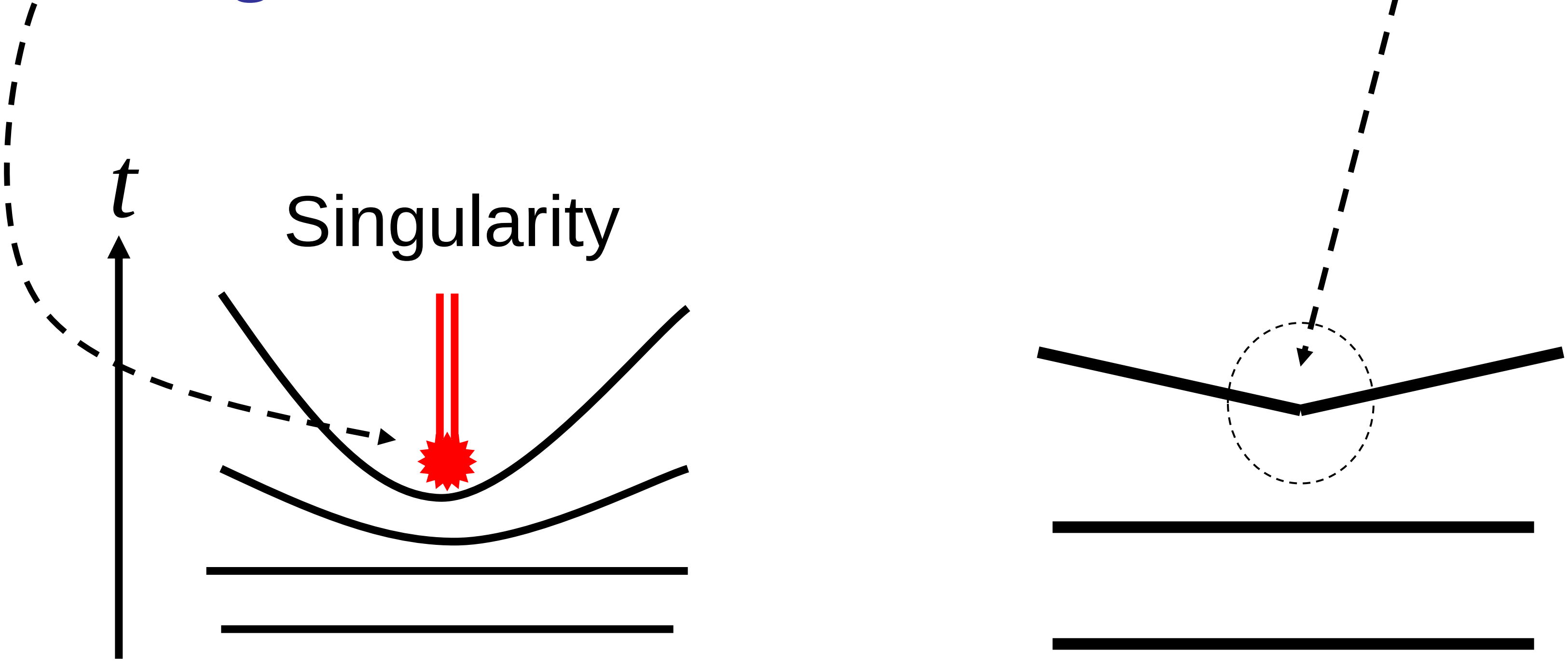
Any time slice and any time axis can be chosen in numerical relativity





Slice: Required properties

– Black hole singularities and coordinate singularities have to be avoided

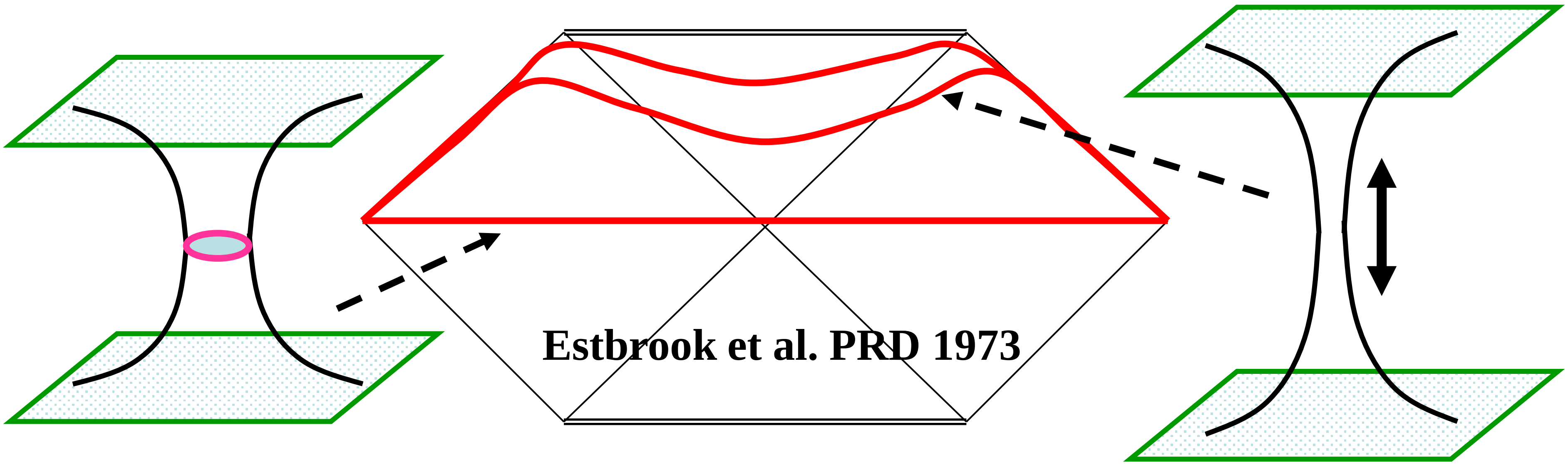


Or, effectively excised

Well-known good slice

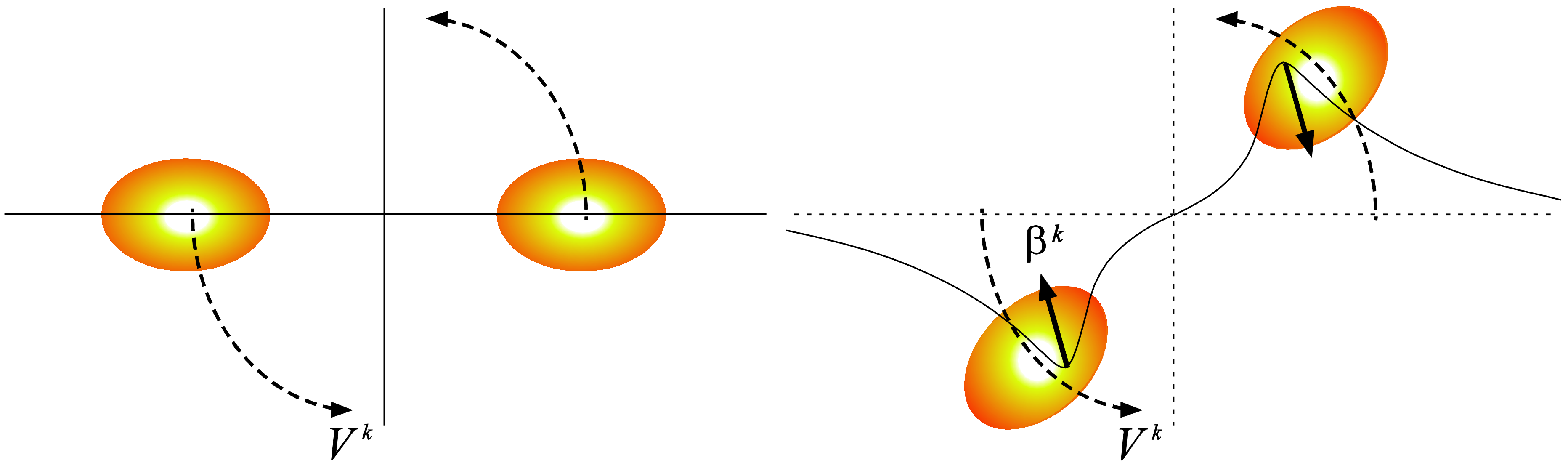
Maximal slice:

$$K = 0 = \dot{K} \Rightarrow \Delta\alpha = \alpha S\left[\gamma_{ij}, K_{ij}, T_{\mu\nu}\right]$$



Drawback: Solving elliptic-type equation is computationally expensive

Spatial gauge; required property



Frame dragging ⇒ coordinates distort

Coordinate shear has to be suppressed.

Minimal distortion gauge (Smarr-York, PRD1978) :

Minimize global distortion defined as

$$I = \int dV (\partial_t \tilde{\gamma}_{ij})(\partial_t \tilde{\gamma}_{kl}) \tilde{\gamma}^{ik} \tilde{\gamma}^{jl} \gamma^{1/2}$$

$$\frac{\partial I}{\partial \beta^i} = 0 \Rightarrow \tilde{D}^k \left(\gamma^{1/2} \partial_t \tilde{\gamma}_{ik} \right) = 0$$

$$\Rightarrow \tilde{\gamma}_{jk} \tilde{\Delta} \beta^k + \frac{1}{3} \tilde{D}_j \tilde{D}_k \beta^k + \tilde{R}_{jk} \beta^k$$

$$+ \tilde{\gamma}_{jl} \tilde{D}_k \ln \gamma^{1/2} \left(\tilde{D}^k \beta^l + \tilde{D}^l \beta^k - \frac{2}{3} \tilde{\gamma}^{kl} \tilde{D}_m \beta^m \right) = S_j$$

Very physical & beautiful.
But, elliptic eqs. & expensive

Puncture gauge; α=0 at puncture

$$\left(\partial_t - \beta^l \partial_l\right)\alpha = -\eta_\alpha \alpha K;$$

$$\eta_\alpha = 2 \text{ for 4D } \& \ 1 < \eta_\alpha < 2 \text{ for N-D}$$

$$\partial_t \beta^k = \frac{D-1}{2(D-2)} V_\beta^2 \gamma^{kl} \left(F_l + \Delta t \partial_t F_l\right)$$

or

$$\begin{cases} \left(\partial_t - \beta^l \partial_l\right)\beta^k = \dfrac{D-1}{2(D-2)} V_\beta^2 B^k \xrightarrow{D=4, V=1} \dfrac{3}{4} B^k \\ \left(\partial_t - \beta^l \partial_l\right) B^k = \left(\partial_t - \beta^l \partial_l\right)\Gamma^k - \eta_\beta B^k \end{cases}$$

$V_\beta^2 = 1$ for 4D and $V_\beta^2 < 1$ is preferable for N-D

$\eta_\beta \sim 1/M_{\rm BH}$ (Alcubierre, Bruegman 03, and others)

Properties

- Slice: Asymptotically approaches to maximal slice-like slice;
  slice freezes ($\alpha_{,t} = 0$, and thus, $K \rightarrow$ const)
- Shift: Similar to minimal distortion gauge !!
- Computational costs are very small
- Horizon sucking does not matter because it happens inside horizon and hyperbolic gauge is suitable for this "effective excision" (Bruegmann et al., 2006)

Section IV: Initial value problem; How to impose constraints

$$ {}^{(3)}R - K_{ij}K^{ij} + K^2 = 16\pi T_{\mu\nu} n^{\mu} n^{\nu} \equiv 16\pi\rho_H $$

$$ D_i K^i_{\ j} - D_j K = -8\pi T_{\mu\nu} n^{\mu} \gamma^{\nu}_{\ j} \equiv 8\pi J_j $$

Only 4 components equations
for 12 component of ***γ*** & ***K***

These are not hyperbolic equations
→ **Write in elliptic equations**
(Method of O'Murchadha-York, 1973)

Hamiltonian constraint

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$$

$$\Rightarrow \quad {}^{(3)}R = \frac{1}{\psi^4}\left(\tilde{R} - \frac{8}{\psi}\tilde{\Delta}\psi \right)$$

See Appendix

$${}^{(3)}R - K_{ij}K^{ij} + K^2 = 16\pi\rho_H$$

$$\Rightarrow \quad \tilde{\Delta}\psi = \frac{\psi}{8}\tilde{R} - 2\pi\rho_H\psi^5 - \frac{\psi^5}{8}\left(K_{ij}K^{ij} - K^2 \right)$$

Elliptic equation for a given set of

$$\left(\tilde{\gamma}_{ij},\ K_{ij},\ \rho_H \right)$$

Momentum constraint

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}, \quad K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij} K$$

$$D_j K_i^{\ j} - D_i K = \psi^{-6} \tilde{D}_j \left(\psi^6 A_i^{\ j} \right) - \frac{2}{3} \tilde{D}_i K \qquad \left(\tilde{D}_i \tilde{\gamma}_{jk} = 0 \right)$$

Then, set

$$\psi^6 A_i^{\ j} = \tilde{D}_i W^j + \tilde{D}^j W_i - \frac{2}{3} \tilde{\gamma}_i^{\ j} \tilde{D}_k W^k + K^{\mathrm{TT}\ j}_{\ \ i}$$

$$\left(\tilde{D}^i K^{TT\ j}_{\ \ i} = 0 = K^{TT\ i}_{\ \ i} \right)$$

$$\Rightarrow \quad \tilde{\Delta} W_i + \frac{1}{3} \tilde{D}_i \tilde{D}_j W^j + \tilde{R}_i^{\ j} W_j - \frac{2}{3} \psi^6 \tilde{D}_i K = 8\pi J_i \psi^6$$

Vector elliptic equation of W_i

for a given set of $\left(\tilde{\gamma}_{ij},\ \psi, K,\ J_j \right)$

Section V: Numerical implementation

- Discretize all variables;

  $\gamma(x, y, z) \rightarrow \gamma_{i, j, k}$ $(i, j, k = 1$—n)

- And then, perform finite difference;

  e.g.,

2nd order finite difference

$$\partial_x \gamma_{i,j,k} = \frac{1}{2\Delta}\left(\gamma_{i+1,j,k} - \gamma_{i-1,j,k}\right)$$

4th order finite difference

$$\partial_x \gamma_{i,j,k} = \frac{1}{12\Delta}\left[-\left\{\gamma_{i-2,j,k} - \gamma_{i+2,j,k}\right\} + 8\left\{\gamma_{i+1,j,k} - \gamma_{i-1,j,k}\right\}\right]$$

Basic equations

$$\partial_t \begin{pmatrix} \tilde{\gamma}_{ij} \\ \chi \\ \tilde{A}_{ij} \\ K \\ \Gamma^i \\ \alpha \\ \beta^i \\ B^i \end{pmatrix} = \beta^k \partial_k \begin{pmatrix} \tilde{\gamma}_{ij} \\ \chi \\ \tilde{A}_{ij} \\ K \\ \Gamma^i \\ \alpha \\ \beta^i \\ B^i \end{pmatrix} + \begin{pmatrix} S_\gamma\left[\tilde{\gamma}_{ij}, \tilde{A}_{ij}, \alpha, \beta^i\right] \\ S_\chi\left[\chi, K, \alpha, \beta^i\right] \\ S_A\left[\tilde{\gamma}_{ij}, \chi, \tilde{A}_{ij}, K, \Gamma^i, \alpha, \beta^i\right] \\ S_K\left[\tilde{\gamma}_{ij}, \chi, \tilde{A}_{ij}, K, \alpha\right] \\ S_\Gamma\left[\tilde{\gamma}_{ij}, \chi, \tilde{A}_{ij}, K, \Gamma^i, \alpha, \beta^i\right] \\ S_\alpha\left[K, \alpha\right] \\ S_\beta\left[B^i\right] \\ S_B\left[\tilde{\gamma}_{ij}, \chi, \tilde{A}_{ij}, K, \Gamma^i, \alpha, \beta^i\right] \end{pmatrix}$$

Method for space finite difference

- Metric = smooth, no shock (different from fluid) → Every variable can be expand as the Taylor series

$$\gamma_{ij}\left(x_0 \pm \Delta\right) = \gamma_{ij} \pm \Delta \gamma_{ij}^{'} + \frac{\Delta^2}{2!}\gamma_{ij}^{''} \pm \frac{\Delta^3}{3!}\gamma_{ij}^{'''} + \frac{\Delta^4}{4!}\gamma_{ij}^{''''}$$

$$\gamma_{ij}\left(x_0 \pm 2\Delta\right) = \gamma_{ij} \pm 2\Delta \gamma_{ij}^{'} + \frac{4\Delta^2}{2!}\gamma_{ij}^{''} \pm \frac{8\Delta^3}{3!}\gamma_{ij}^{'''} + \frac{16\Delta^4}{4!}\gamma_{ij}^{''''}$$

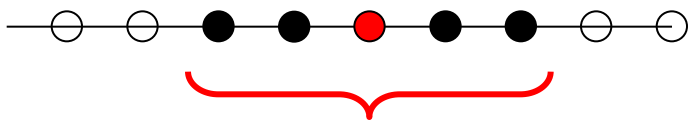
4th order scheme

4 derivatives ←→ 5 values

Need 1st and 2nd derivatives

$$\partial_x \gamma_{ab} = \frac{1}{12\Delta}\Big[-\{\gamma_{ab,j+2,k,l} - \gamma_{ab,j-2,k,l}\} + 8\{\gamma_{ab,j+1,k,l} - \gamma_{ab,j-1.k.l}\}\}\Big]$$

$$\partial_x^2 \gamma_{ab} = \frac{1}{12\Delta^2}\Big[-\{\gamma_{ab,j+2,k,l} + \gamma_{ab,j-2,k,l}\} + 16\{\gamma_{ab,j+1,k,l} + \gamma_{ab,j-1,k,l}\} - 30\gamma_{ab,j,k,l}\Big]$$



Write all the derivatives using these rules:

Currently, 4th-order is most popular

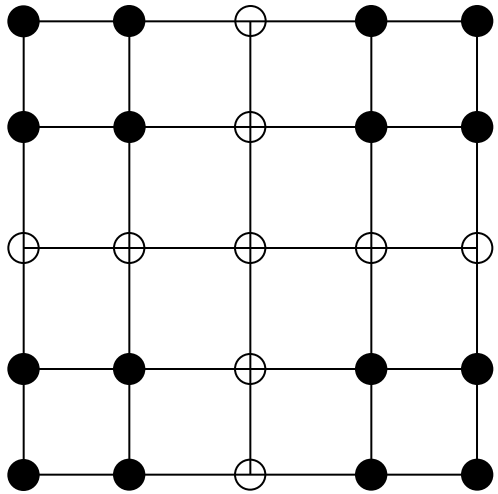
Einstein's equation is complicated

$$\partial_x \partial_y \gamma_{ab} = \left(\frac{1}{12\Delta}\right)^2 \Big[-\Big\{ -\left(\gamma_{ab,i+2,j+2,k} - \gamma_{ab,i-2,j+2,k}\right) + 8\left(\gamma_{ab,i+1,j+2,k} - \gamma_{ab,i-1,j+2,k}\right)\Big\}$$

$$+\Big\{ -\left(\gamma_{ab,i+2,j-2,k} - \gamma_{ab,i-2,j-2,k}\right) + 8\left(\gamma_{ab,i+1,j-2,k} - \gamma_{ab,i-1,j-2,k}\right)\Big\}$$

$$+8\Big\{ -\left(\gamma_{ab,i+2,j+1,k} - \gamma_{ab,i-2,j+1,k}\right) + 8\left(\gamma_{ab,i+1,j+1,k} - \gamma_{ab,i-1,j+1,k}\right)\Big\}$$

$$-8\Big\{ -\left(\gamma_{ab,i+2,j-1,k} - \gamma_{ab,i-2,j-1,k}\right) + 8\left(\gamma_{ab,i+1,j-1,k} - \gamma_{ab,i-1,j-1,k}\right)\Big\}\Big]$$



Many points are used

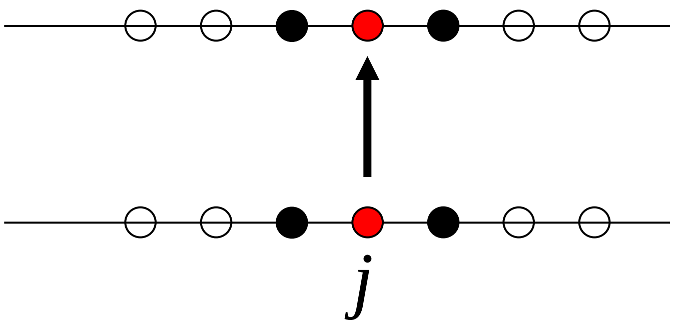
Advection term: Upwind scheme I

Consider the simplest wave equation:

$$\partial_t f + c\partial_x f = 0 \quad \Rightarrow \quad f = F\left(x - ct\right)$$

Simple Finite difference

$$f_j^{n+1} = f_j^n + \frac{c\Delta t}{\Delta x}\left(f_{j+1}^n - f_{j-1}^n \right)$$



$\Rightarrow$ unstable !!

$\because$ von Neumann stability analysis

$$u_j^n = \xi^n \exp\left(ikx_j\right) = \xi^n \exp\left(ikj\Delta x\right)$$

$$\Rightarrow \quad \xi = 1 - i\frac{c\Delta t}{\Delta x}\sin\left(k\Delta x\right) \quad \Rightarrow \quad \left|\xi\right| > 1$$

Cf. Numerical Recipe (Press et al.)

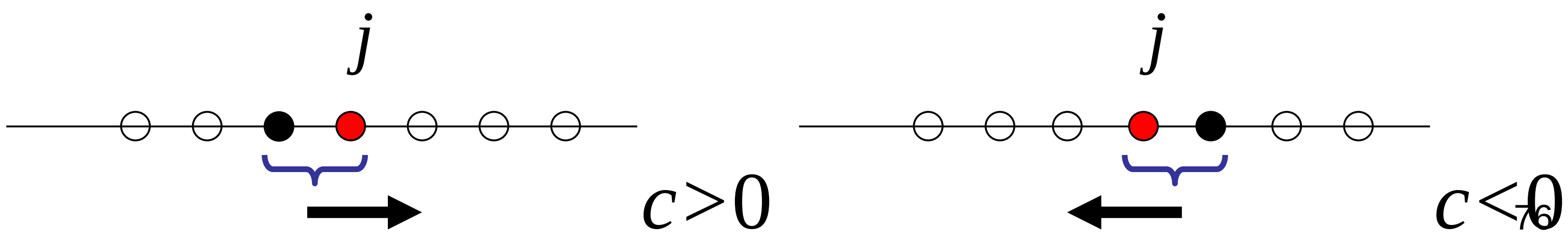
Advection term: Upwind scheme II

Simplest stable method

= First-order upwind scheme

$$f_j^{n+1} = \begin{cases} f_j^n + \dfrac{c\Delta t}{\Delta x}\left(f_j^n - f_{j-1}^n\right) & c > 0 \\ f_j^n + \dfrac{c\Delta t}{\Delta x}\left(f_{j+1}^n - f_j^n\right) & c < 0 \end{cases}$$

$\because$ von Neumann stability analysis $\Rightarrow |\xi| \leq 1$

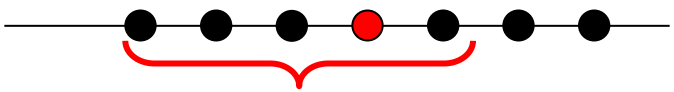


Advection term: Upwind scheme III

Advection term: $\beta^l \partial_l \gamma_{ij}$ 4th-order scheme

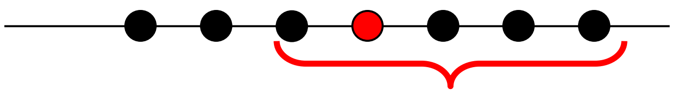
$\Rightarrow$ Need to use upwind scheme

For $-\beta^l > 0$,



$$\gamma'_{ij} = \frac{1}{12\Delta}\Big[-\gamma_{ij}(x_0 - 3\Delta) + 6\gamma_{ij}(x_0 - 2\Delta) - 18\gamma_{ij}(x_0 - \Delta) + 10\gamma_{ij}(x_0) + 3\gamma_{ij}(x_0 + \Delta)\Big]$$

For $-\beta^l < 0$,



$$\gamma'_{ij} = -\frac{1}{12\Delta}\Big[-\gamma_{ij}(x_0 + 3\Delta) + 6\gamma_{ij}(x_0 + 2\Delta) - 18\gamma_{ij}(x_0 + \Delta) + 10\gamma_{ij}(x_0) + 3\gamma_{ij}(x_0 - \Delta)\Big]$$

Time evolution: Standard one

- Use Runge-Kutta method
- Simple 2nd order method → Unstable
- 3rd or 4th order (popular) → Stable

$$\partial_t Q_a = F\left[Q_a\right]$$

$$\begin{cases} k_1 = \Delta t F\left[Q_a\left(t_0\right)\right] \\ k_2 = \Delta t F\left[Q_a\left(t_0\right) + k_1/2\right] \\ k_3 = \Delta t F\left[Q_a\left(t_0\right) + k_2/2\right] \\ k_4 = \Delta t F\left[Q_a\left(t_0\right) + k_3\right] \end{cases}$$

$$Q_a\left(t_0 + \Delta t\right) = Q_a\left(t_0\right) + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} + O\left(\Delta t^5\right)$$

Boundary conditions

- Geometry obeys wave equations

Outgoing boundary condition

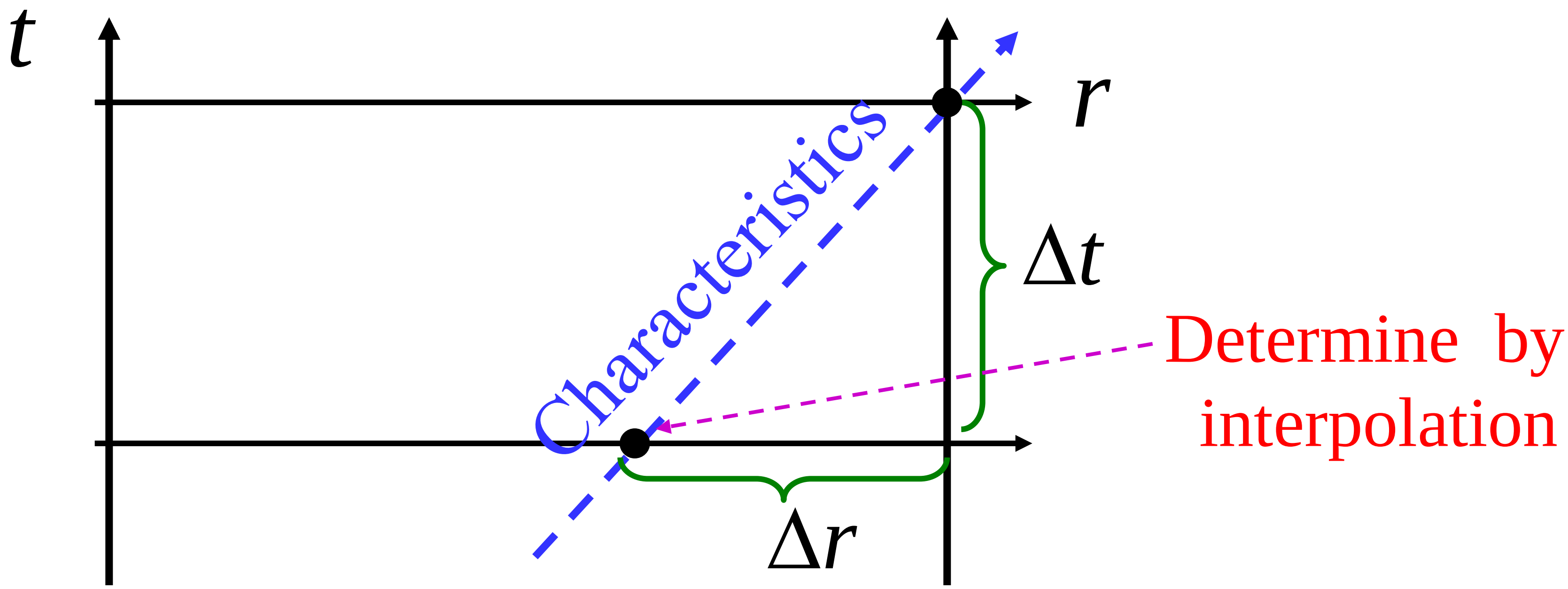
$$Q_a \to \frac{F_a\left(t - r/c\right)}{r} + O\left(1/r^2\omega\right)$$

ω: characteristic angular frequency

Usually, only the leading order is taken into account.

Location of boundary should be $r\omega \gg 1$

Numerical implementation of outgoing boundary condition



$$rQ_a = F_a\left(t - r/c\right)$$

$$\Rightarrow \quad rQ_a\left(t - r/c\right)$$

$$= \left(r - \Delta r\right) Q_a \left(t - \Delta t - (r - \Delta r)/c\right)$$

Section: Extracting gravitational waves by complex Weyl scalar

Using a complex Weyl scalar

$$\Psi_4 \equiv -R_{\alpha\beta\gamma\delta}\tilde{n}^{\alpha}\bar{m}^{\beta}\tilde{n}^{\gamma}\bar{m}^{\delta}$$

$R_{\alpha\beta\gamma\delta}$: 4D Riemann tensor

$\left(\tilde{n}^{\alpha}, \tilde{l}^{\alpha}, m^{\alpha}, \bar{m}^{\alpha}\right)$: Null tetrad

$$r \to \infty,\ \ \Psi_4 \to -\frac{1}{2}\left(\ddot{h}_{+} - i\ddot{h}_{\times}\right)$$

$$\Rightarrow\ \ h_{+} - ih_{\times} = -2\int_{t} dt' \int_{t'} dt'' \Psi_4\left(t''\right)$$

Weyl scalar for gravitational waves I

$\left(\tilde{n}^{\alpha}, \tilde{l}^{\alpha}, m^{\alpha}, \bar{m}^{\alpha}\right)$: Null tetrad

$$\tilde{n}^{\alpha}\tilde{n}_{\alpha} = \tilde{l}^{\alpha}\tilde{l}_{\alpha} = m^{\alpha}m_{\alpha} = 0, \quad -\tilde{n}^{\alpha}\tilde{l}_{\alpha} = 1 = m^{\alpha}\bar{m}_{\alpha}$$

$$g_{\mu\nu} = -\tilde{n}_{\mu}\tilde{l}_{\nu} - \tilde{l}_{\mu}\tilde{n}_{\nu} + m_{\mu}\bar{m}_{\nu} + \bar{m}_{\mu}m_{\nu}$$

$\tilde{n}^{\mu}$: Ingoing Null $\rightarrow \dfrac{1}{\sqrt{2}}\left(n^{\mu} - r^{\mu}\right)$

m^{μ} : Complex orthnormal to $\tilde{n}^{\alpha}$ & $\tilde{l}^{\alpha}$

n^{μ} : Unit timelike normal as before

r^{μ} : Unit radial vector, $r^{\mu}n_{\mu} = 0$

Weyl scalar for gravitational waves II

$$\Psi_4 = -\frac{1}{2}\left({}^{(4)}R_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta - 2\,{}^{(4)}R_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta r^\gamma \bar{m}^\delta \right.$$

$$\left. + {}^{(4)}R_{\alpha\beta\gamma\delta} r^\alpha \bar{m}^\beta r^\gamma \bar{m}^\delta \right)$$

$${}^{(4)}R_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta = \left(R_{ij} - K_i^k K_{jk} + K K_{ij} \right) \bar{m}^i \bar{m}^j \equiv E_{ij} \bar{m}^i \bar{m}^j$$

$${}^{(4)}R_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta r^\gamma \bar{m}^\delta = \left(D_j K_{ik} - D_k K_{ij} \right) \bar{m}^i r^j \bar{m}^k \equiv B_{ijk} \bar{m}^i r^j \bar{m}^k$$

$${}^{(4)}R_{\alpha\beta\gamma\delta} r^\alpha \bar{m}^\beta r^\gamma \bar{m}^\delta = \left(R_{ijkl} + K_{ik} K_{jl} - K_{il} K_{jk} \right) r^i \bar{m}^j r^k \bar{m}^l$$

cf, derivation of Gauss-Codacci eqs.

Note in 3D,

$$R_{ijkl} = \gamma_{ik} R_{jl} - \gamma_{il} R_{jk} - \gamma_{jk} R_{il} + \gamma_{jl} R_{ik} - \frac{R}{2}\left(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk} \right)$$

Weyl scalar for gravitational waves III

$$\mathcal{R}_{ijkl} = R_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk}$$

$$\mathcal{R}_{ik} = R_{ik} + K_{ik}K - K_{il}K^l_k = E_{ik},$$

$E = R + K^2 - K_i^j K_j^i = 0$ (vacuum, Hamiltonian constr.)

$$\therefore \quad \mathcal{R}_{ijkl} = \gamma_{ik}E_{jl} - \gamma_{il}E_{jk} - \gamma_{jk}E_{il} + \gamma_{jl}E_{ik} - \frac{E}{2}\left(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}\right)$$

& then $\mathcal{R}_{ijkl} r^i \bar{m}^j r^k \bar{m}^l = E_{jl}\bar{m}^j \bar{m}^l$

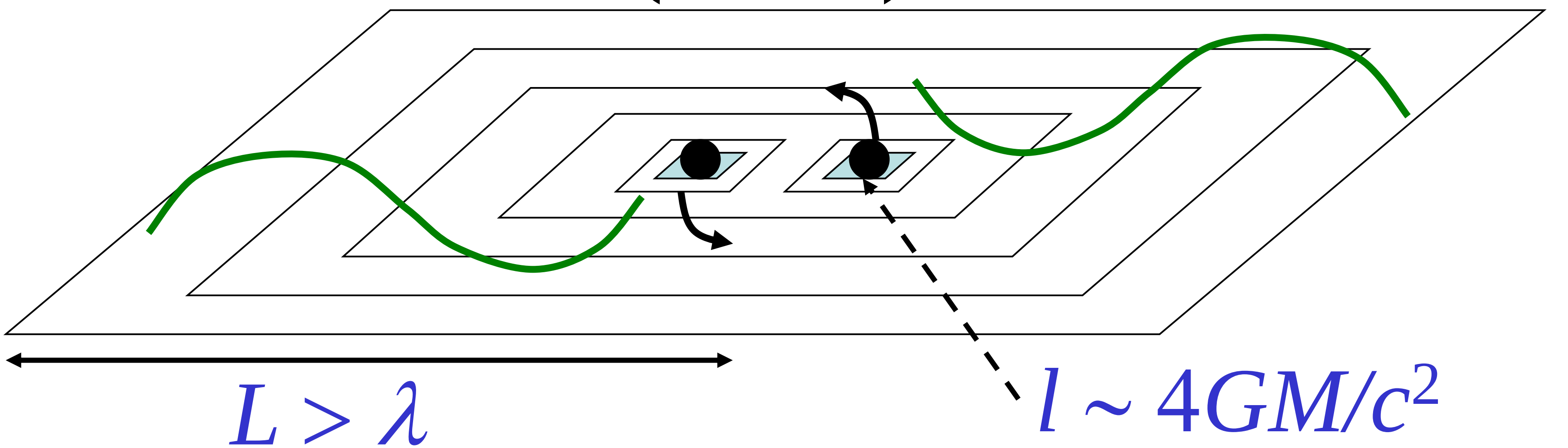
where $\gamma_{jl}\bar{m}^j \bar{m}^l = 0, \quad \gamma_{jl} r^j \bar{m}^l = 0, \quad \gamma_{jl} r^j r^l = 1$

Finally,

$$\Psi_4 = -\left(E_{ik}\bar{m}^i \bar{m}^k - B_{ijk}\bar{m}^i r^j \bar{m}^k\right)$$

Quite simple

Section: Adaptive Mesh Refinement (AMR)



Why is it required ?

- More than 2 different length scales
- For binary, gravitational wavelength and radius of compact star

→ Resolve stars while extracting waves

Typical length scales

Radius of black hole $R \approx \dfrac{Gm}{c^2}$

Radius of neutron star $R \approx (5\sim 8)\dfrac{Gm}{c^2}$

Angular velocity of binary $\Omega \approx \sqrt{\dfrac{GM}{r^3}}, \; M = m_1 + m_2$

Gravitational wavelength $\lambda = \dfrac{\pi c}{\Omega}$

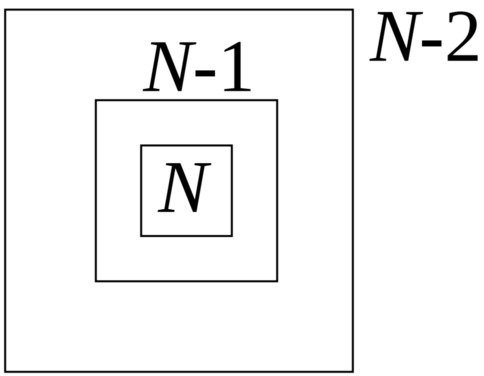
$$\Rightarrow \quad \lambda \approx \pi c \sqrt{\frac{r^3}{GM}} \approx 100 \left(\frac{r}{10GM/c^2} \right)^{3/2} \left(\frac{GM}{c^2} \right) \square \;\; R$$

r is orbital radius $\geq \sim 5\dfrac{GM}{c^2}$

AMR grid in numerical relativity

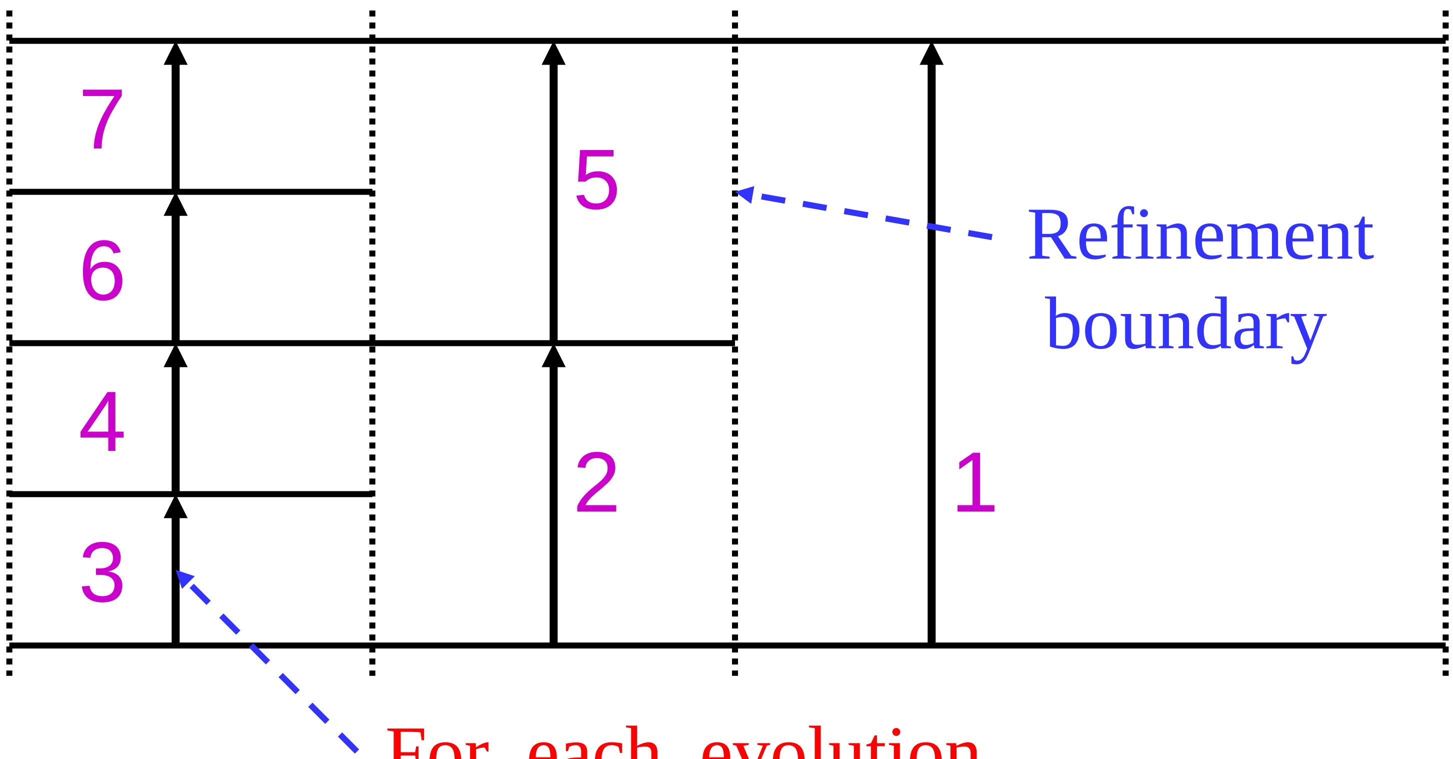
- Prepare N=5—10 levels of different resolutions and domain size: Domain D_l, Size L_l, Grid spacing Δ_l where l=1—N
- For the simple case, we set $2L_l = L_{l\text{-}1}$, $2\Delta_l = \Delta_{l\text{-}1}$ (same grid number for l=1—N) and $L_1 > \lambda,\ \Delta_N << Gm/c^2$

Ratio=2:1



AMR time step

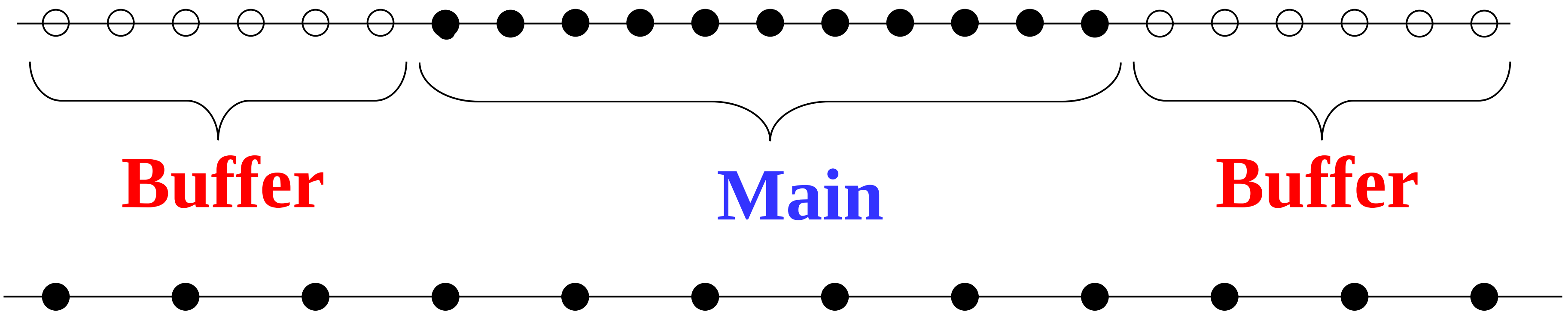
- For the finer grid spacing, the time step is smaller: $2\Delta t_l = \Delta t_{l-1}$



For each evolution,
4th-order Runge-Kutta method is used

Key: Interpolation and treatment of buffer zone

- Grid in each domain is composed of main and buffer zones

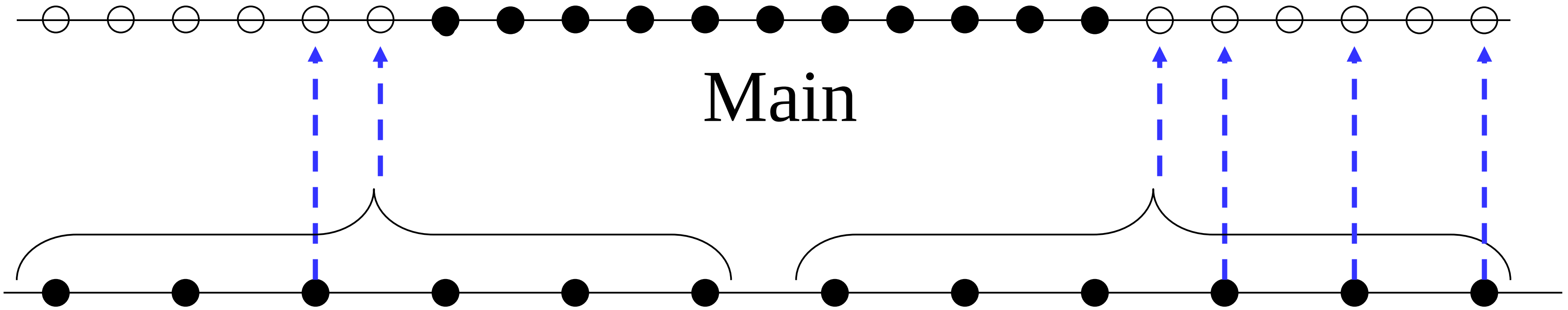


Main : Solve equations with no prescription

Buffer: Interpolation is necessary

A method for interpolation: Example I

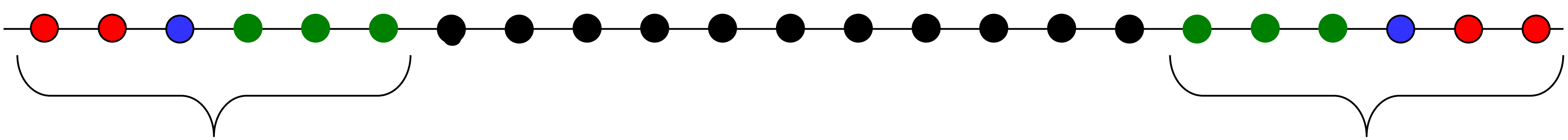
- Before the first Runge-Kutta step → Quantity in the buffer zone is determined by Lagrange interpolation, e.g., 5th-order interpolation (using 6 points)



For 3-dimension, 6^3=216 grid points are used

A method for interpolation II

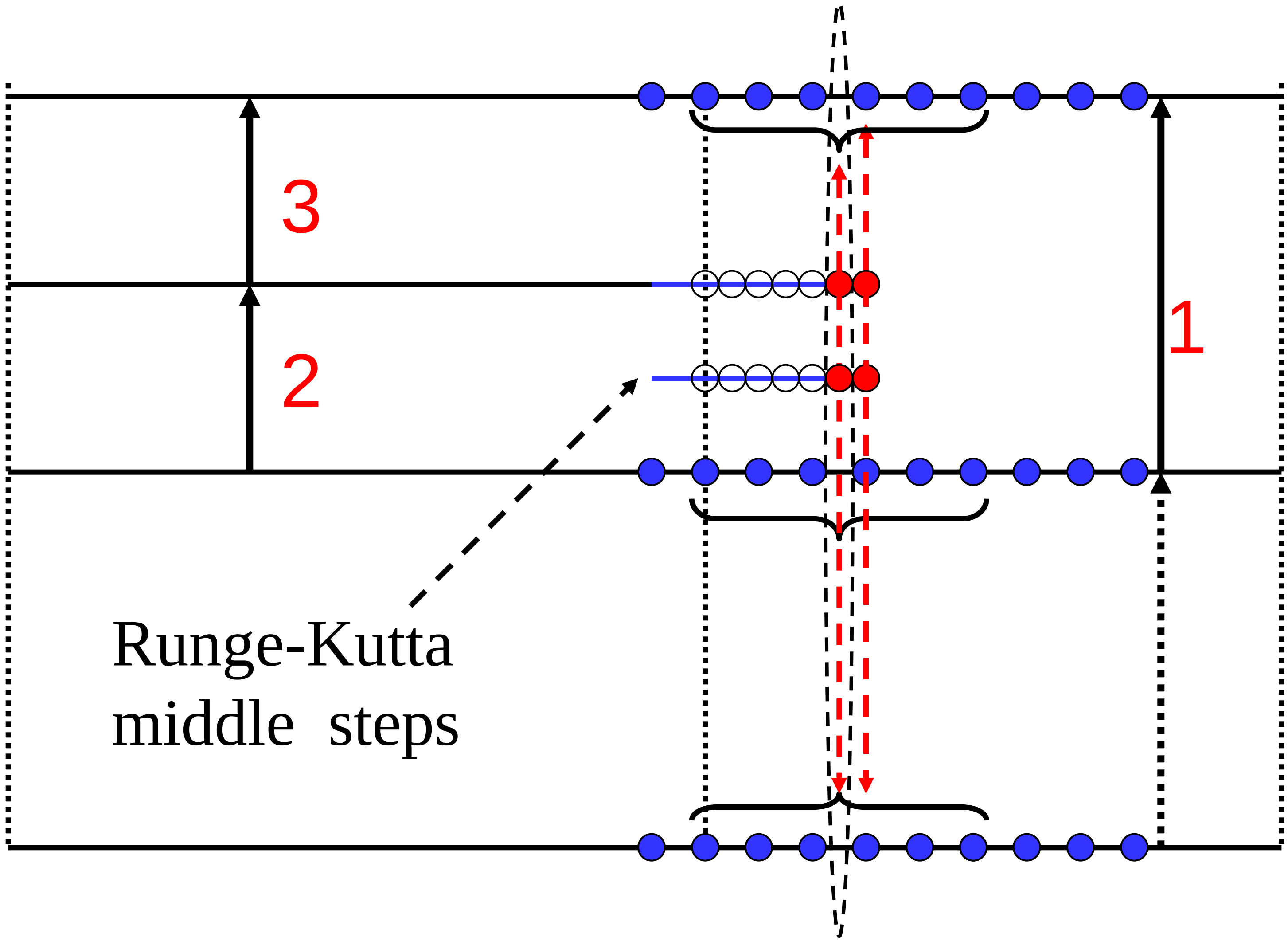
- The Runge-Kutta evolution step
- Inner 3 buffer points are solved in the same manner as in main region
- 4^{th} one: The same as above but for advection term; mixture of 2^{nd} and 4^{th} order upwind schemes
- 5^{th} and 6^{th} one: Interpolation in *space and time*: Time = 2^{nd}-order interpolation



Bruegmann et al., Yamamoto et al., PRD2008

AMR time interpolation

- Concept of time interpolation



Moving domain

- Small-size domains (finer domains) which cover compact objects move with them
- Large-size domains (coarser domains) do not move (fixed grid)
- Finer domains are moved when the center of black hole or neutron star moves

For black holes, solve $\dfrac{dx^i}{dt} = -\beta^i$

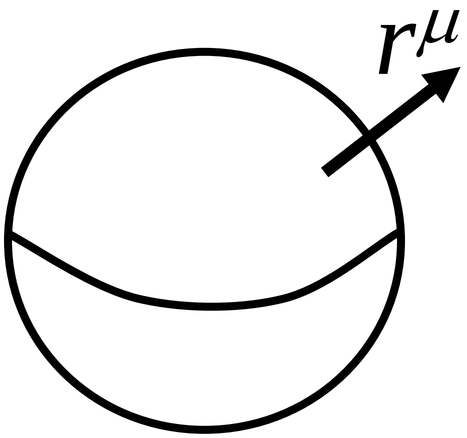
$$\left(\because \quad \left(\partial_t - \beta^k \partial_k\right)\alpha = 0 \text{ at black hole center, } \alpha = 0 \text{ in the puncture gauge} \right)$$

For neutron stars, find the location of max density

Section: Finding a black hole

- Popular method = **Finding apparent horizon**
- Apparent horizon = surface for which *expansion of outgoing null ray is zero*
- Important property (Hawking-Ellis, 1973): *If an apparent horizon exists for a globally hyperbolic spacetime, event horizon of a black hole always exists*
  → **Apparent horizon is formed**
  = A black hole is formed

Basic equation for AH I



- Define outgoing null vector

$l^{\mu} = n^{\mu} + r^{\mu}$

n^{μ} : Time like unit normal

r^{μ} : Radial unit normal, $r^{\mu} n_{\mu} = 0.$

r^{μ} is perpendicular to AH sphere.

a^{μ}, b^{μ} : Unit normal on sphere

$$\Rightarrow \quad g_{\mu\nu} = -n_{\mu} n_{\nu} + r_{\mu} r_{\nu} + a_{\mu} a_{\nu} + b_{\mu} b_{\nu}$$

Basic equation for AH II

- Expansion = change rate of area

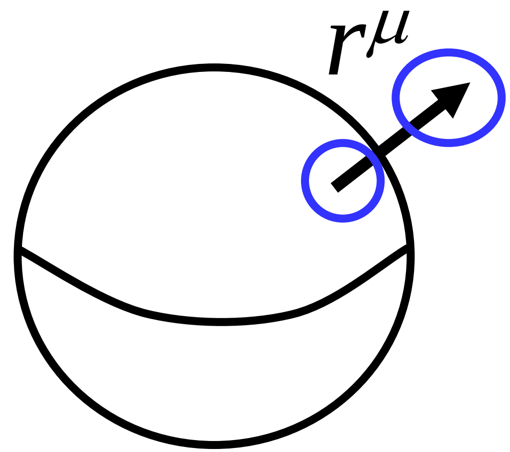
$$\Theta \equiv \left(a^\mu a^\nu + b^\mu b^\nu\right)\nabla_\mu l_\nu$$

$$= \left(\gamma^{\mu\nu} - r^\mu r^\nu\right)\nabla_\mu\left(n_\nu + r_\nu\right)$$

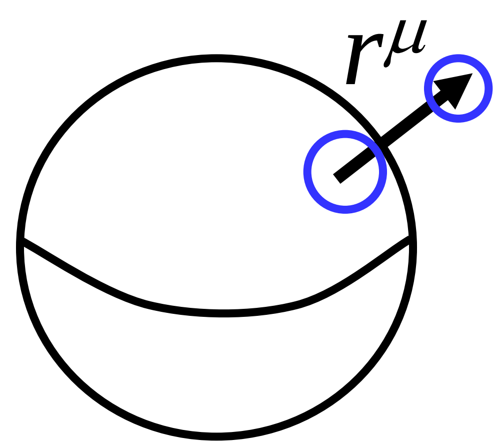
$$= -K + D_k r^k + r^i r^j K_{ij}$$

Apparent horizon: $\Theta = 0$

$$\Rightarrow \quad D_k r^k + r^i r^j K_{ij} - K = 0$$



Outside AH



Inside AH

Basic equation for AH III

- AH=two sphere, $r=h(\theta, \phi)$

$$r_i = C\left(1,\ -\partial_\theta h,\ -\partial_\varphi h\right)$$

C is determined by $r_i r^i = 1$

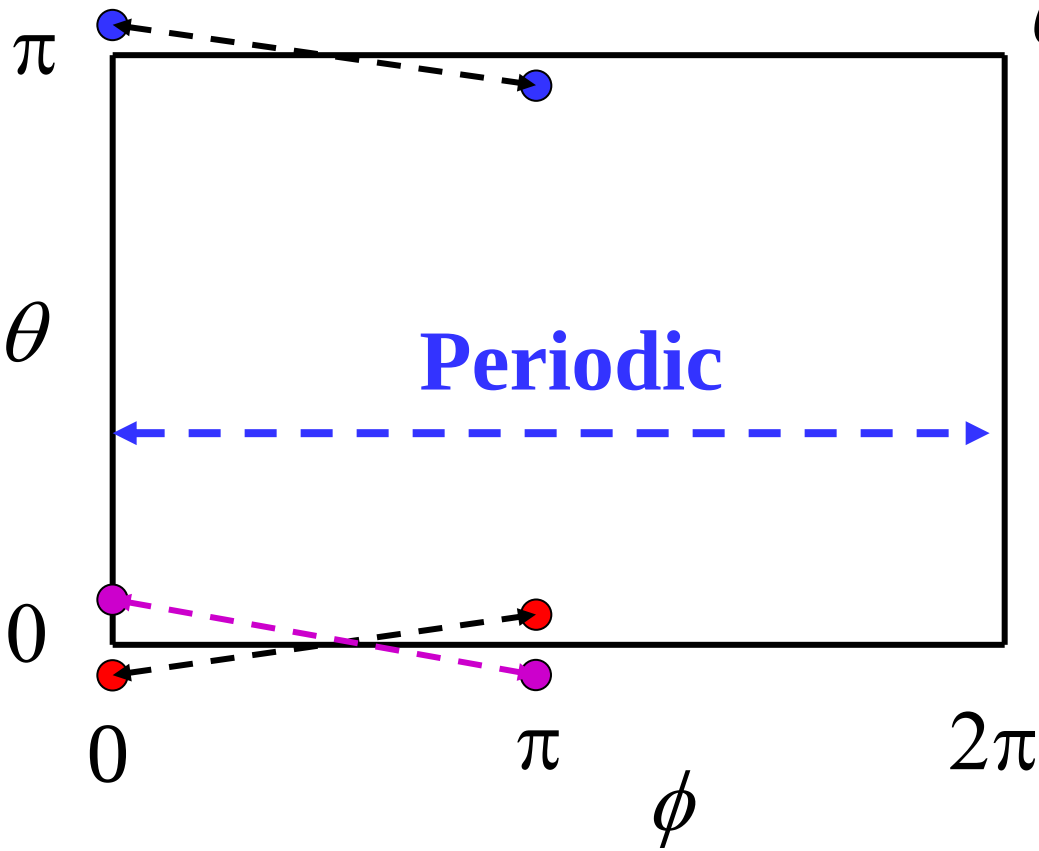
$$\Rightarrow \quad D_k r^k = \frac{1}{\sqrt{\gamma}}\partial_k\left(\sqrt{\gamma}r^k\right)$$

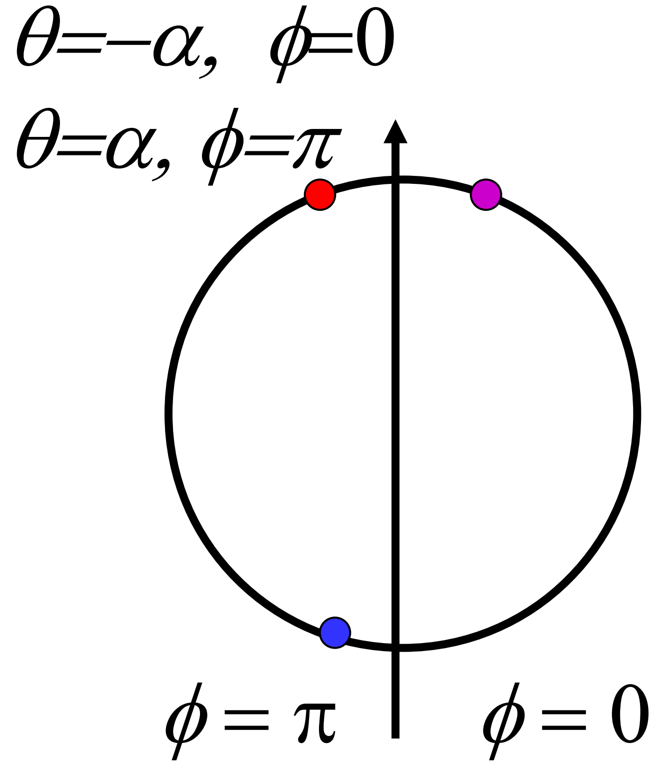
$$= C\left(-\partial_\theta^2 h - \cot\theta\partial_\theta h - \frac{1}{\sin^2\theta}\partial_\varphi^2 h\right) + \ldots.$$

$\Rightarrow \quad \Theta = 0 \quad \cdots \quad$ 2D elliptic type equation

Basic equation for AH IV

- Boundary condition





Well defined → Standard solver is OK

Appendix: Conformal decomposition I

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}, \quad D_i \gamma_{jk} = 0, \quad \tilde{D}_i \tilde{\gamma}_{jk} = 0$$

$$\Gamma^i_{jk} = \frac{1}{2}\gamma^{il}\left(\partial_j \gamma_{kl} + \partial_k \gamma_{jl} - \partial_l \gamma_{jk}\right)$$

$$= \frac{1}{2}\tilde{\gamma}^{il}\left(\partial_j \tilde{\gamma}_{kl} + \partial_k \tilde{\gamma}_{jl} - \partial_l \tilde{\gamma}_{jk}\right)$$

$$+ \frac{2}{\psi}\tilde{\gamma}^{il}\left(\tilde{\gamma}_{kl}\partial_j \psi + \tilde{\gamma}_{jl}\partial_k \psi - \tilde{\gamma}_{jk}\partial_l \psi\right)$$

$$= \tilde{\Gamma}^i_{jk} + \frac{2}{\psi}\left(\delta^i_k \tilde{D}_j \psi + \delta^i_j \tilde{D}_k \psi - \tilde{\gamma}_{jk}\tilde{D}^i \psi\right)$$

$$= \tilde{\Gamma}^i_{jk} + C^i_{jk}$$

Conformal decomposition II

$$R_{jk} = \partial_i \Gamma^i_{jk} - \partial_j \Gamma^i_{ik} + \Gamma^i_{jk}\Gamma^l_{il} - \Gamma^i_{jl}\Gamma^l_{ik}$$

$$= \partial_i \tilde{\Gamma}^i_{jk} - \partial_j \tilde{\Gamma}^i_{ik} + \tilde{\Gamma}^i_{jk}\tilde{\Gamma}^l_{il} - \tilde{\Gamma}^i_{jl}\tilde{\Gamma}^l_{ik}$$

$$+ \partial_i C^i_{jk} - \partial_j C^i_{ik} + C^i_{jk} C^l_{il} - C^i_{jl} C^l_{ik}$$

$$+ \tilde{\Gamma}^i_{jk} C^l_{il} - \tilde{\Gamma}^i_{jl} C^l_{ik} + C^i_{jk}\tilde{\Gamma}^l_{il} - C^i_{jl}\tilde{\Gamma}^l_{ik}$$

$$= \tilde{R}_{jk} + \tilde{D}_i C^i_{jk} - \tilde{D}_j C^i_{ik} + C^i_{jk} C^l_{il} - C^i_{jl} C^l_{ik}$$

$$R^{\psi}_{jk} = \tilde{D}_i C^i_{jk} - \tilde{D}_j C^i_{ik} + C^i_{jk} C^l_{il} - C^i_{jl} C^l_{ik}$$

$$= \frac{1}{\psi^2}\left(6\tilde{D}_j\psi\tilde{D}_k\psi - 2\tilde{\gamma}_{jk}\tilde{D}_l\psi\tilde{D}^l\psi\right)$$

$$-\frac{2}{\psi}\left(\tilde{D}_j\tilde{D}_k\psi + \tilde{\gamma}_{jk}\tilde{\Delta}\psi\right)$$

Methods for code validation

- **Confirm to reproduce exact solutions:**
  -- Black hole solutions (check area etc)
  -- Propagation of linear gravitational waves
- **Check violation of constraints is small:**
  Monitor Hamiltonian and momentum constraints
- **Check convergence:**
  If 4^{th}-order scheme is used, quantities converges at 4^{th} order; $Q = Q_0 + Q_4 \Delta x^4$

Some technical issues I

- Imposing some constraints within O(Δ^4):

$$\det\left(\tilde{\gamma}_{ij}\right)=1 \quad \text{and} \quad \tilde{A}_{ij}\tilde{\gamma}_{ij}=0$$

$$\Rightarrow \begin{cases} \tilde{\gamma}_{ij} \rightarrow \left[\det\left(\tilde{\gamma}_{ij}\right)\right]^{-1/3}\tilde{\gamma}_{ij} \\ \tilde{A}_{ij} \rightarrow \left[\det\left(\tilde{\gamma}_{ij}\right)\right]^{-1/3}\tilde{A}_{ij}-\dfrac{1}{3}\tilde{\gamma}_{ij}\mathrm{Tr}\left(\tilde{A}_{ij}\right) \\ W \rightarrow \left[\det\left(\tilde{\gamma}_{ij}\right)\right]^{-1/6}W \\ K \rightarrow K+\mathrm{Tr}\left(\tilde{A}_{ij}\right) \end{cases}$$

γ_{ij}, K_{ij} are unchanged

Some technical issues II

- Kreiss-Oliger-type dissipation > O(Δ^4):

$$Q \to Q - \sigma \Delta x^6 Q^{(6)}$$

σ is a constant of $\mathrm{O}\left(0.1\right)$

$Q^{(6)}$ is the sum of sixth derivatives

Purpose is to suppress
high-frequency numerical noise