Monte Carlo approach to string/M theory

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Goal (1): understand these plots.

- Energy density
- Two-point function
- Supersymmetric Polyakov loop
- Power law predicted by SUGRA
- Schwarzshild radius
Goal (2)

Make you convince that lattice theorists can give important contributions to string/M-theory, which usual string theorists can never achieve.

inflation, birth of the universe, multiverse, Hawking evapolation...
Plan

(1) Gauge/Gravity duality (AdS/CFT) and Super Yang-Mills

(2) Why lattice SUSY is hard (fine tuning, sign problem)

(3) How to put SYM on computer. (i).
   D0-brane quantum mechanics

(4) How to put SYM on computer. (ii).
   (1+1)-d theories

(5) How to put SYM on computer. (iii).
   (2+1)-d and (3+1)-d theories

(6) ABJM Theory and M-theory (‘membrane mini revolution’)
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(6) ABJM Theory and M-theory (‘membrane mini revolution’)
• What is string theory?
• Supergravity as a low-energy effective theory
• D-brane and second string revolution
• SYM from D-brane
• Gauge/gravity duality
What is string theory?
(Nobody knows the final answer yet ;)

• Point particles are promoted to (1+1)-dimensional ‘string’.

• Open string $\rightarrow$ gauge fields and infinitely many massive fields

• Closed string $\rightarrow$ graviton, tensors and massive fields

• The scattering amplitudes can be calculated.
Supergravity as a low-energy effective theory

- From the scattering amplitudes, one can determine the low-energy effective action in terms of the massless fields, which reproduce the amplitudes.
  
  IIA/IIB superstring ⇒ IIA/IIB supergravity

- There are BPS ‘black p-brane’ solutions coupling to p-form tensor
  
  $(p+1)$-d analogues of the black hole
  
  $(p=\text{even for IIA, } p=\text{odd for IIB})$

Can we understand black p-branes from the perturbative string picture?
black brane = D-brane

• Dp-brane (Dirichlet p-brane) is a (p+1)-d object on which open string can be attached.

• It has the same charge as the black p-brane.

Dirichlet boundary condition along the transverse direction

Polchinski
Massless d.o.f.

\( \Lambda_\mu(x_\mu) : \text{Gauge field} \)

\( \Phi(x_\mu) : \text{Adjoint scalar field} \)

(coordinate of the brane)

(and adjoint fermions)
SYM from D-brane (2)

A_μ and Φ become
N×N matrix ex

(i,j)-component
= string connecting
i-th and j-th D-branes

→ U(N) Super Yang-Mills
(more generally, the Dirac-Born-Infeld action)

bunch of many D-branes (N>>1)
= black brane

(large-N → heavy and big → classical gravity)
Maximally supersymmetric Yang-Mills

(10d $\mathcal{N} = 1$ U(N) SYM)

\[ S_{10d} = \frac{1}{g_{YM}^2} \int d^{10}x Tr \left( \frac{1}{4} F_{MN}^2 + \frac{1}{2} \bar{\psi} \gamma^M D_M \psi \right) \]

\[ S_{(p+1)d} = \frac{1}{g_{YM}^2} \int d^{p+1}x Tr \left( \frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} (D_\mu X_i)^2 + \frac{1}{4} [X_i, X_j] \right. \\
\left. + \frac{1}{2} \bar{\psi} \gamma^\mu D_\mu \psi - \frac{i}{2} \bar{\psi} \gamma^i [X_i, \psi] \right) \]

Dp-brane worldvolume

dimensional reduction

\[ \delta A_M \sim \bar{\epsilon} \Gamma_M \psi \]
\[ \delta \psi \sim F^{MN} \Gamma_{MN} \epsilon \]
Gauge/gravity duality conjecture (Maldacena 1997)

• In a special limit, both SYM and weakly-coupled string pictures become valid.

• But they are two different descriptions of the same D-brane system. So...

$$\text{SYM} = \text{superstring}$$
black p-brane solution

(Horowitz-Strominger 1991)

\[ ds^2 = \alpha' \left\{ \frac{U^{7-p}}{g_{YM} \sqrt{d_p N}} \left[ -\left( 1 - \frac{U_0^{7-p}}{U^{7-p}} \right) dt^2 + \sum_{i=1}^{p} dy_i^2 \right] \right. \\
+ \frac{g_{YM} \sqrt{d_p N}}{U^{7-p/2} \left( 1 - \frac{U_0^{7-p}}{U^{7-p}} \right)} dU^2 + \left. g_{YM} \sqrt{d_p N} U^{p-8} d\Omega_{8-p}^2 \right\}, \]

\[ e^\phi = (2\pi)^{2-p} g_{YM}^2 \left( \frac{g_{YM}^2 d_p \sqrt{N}}{U^{7-p}} \right)^{\frac{3-p}{4}}, \]

\[ d_p = 2^{7-2p} \pi^{\frac{9-3p}{2}} \Gamma \left( \frac{7-p}{2} \right), \]

For \( p=3 \), SUGRA is valid when \( \lambda = g_{YM}^2 N >> 1 \) and \( g_{YM} << 1 \)

\( 1/N \) expansion = \( g_{st} \)-expansion when \( \lambda \) is fixed
The dictionary

**Gravity**
- ADM mass
- minimal surface
- mass of field excitation

**SYM**
- Energy density
- Wilson/Polyakov loop
- scaling dimension
“In principle, we can use this duality to give a definition of M/string theory on flat spacetime as (a region of) the large \( N \) limit of the field theories. Notice that this is a non-perturbative proposal for defining such theories, since the corresponding field theories can, in principle, be defined non-perturbatively.”
Important not only conceptually, but also *practically*.

If Maldacena’s conjecture is correct, by putting super Yang-Mills on computer one can simulate superstring!
SYM

large-\(N\),
strong coupling

large-\(N\),
finite coupling

finite-\(N\),
finite coupling

STRING

SUGRA

tree-level string
(\text{SUGRA}+\alpha')

Quantum string
\((g_{\text{string}}>0)\)
SYM \text{difficult}

large-N, strong coupling

large-N, finite coupling

finite-N, finite coupling

STRING

SUGRA easier

tree-level string (SUGRA+α')
more difficult

Quantum string ($g_{\text{string}} > 0$)
very difficult
SYM_{difficult}

large-N, strong coupling

large-N, finite coupling

finite-N, finite coupling

STRING

SUGRA easier

tree-level string (SUGRA+\alpha') more difficult

Quantum string (g_{string}>0) very difficult
The opposite direction of the dictionary can be useful, if we use Monte Carlo!
Very important remark

- Form the string theory point of view, **SYM theories in less than four spacetime dimensions are as interesting as four dimensional theories!**

  \[(0+1)-d \text{ SYM} \iff \text{Black hole}\]
  \[(1+1)-d \text{ SYM} \iff \text{Black 1-brane, black string}\]
  \[(3+1)-d \text{ SYM} \iff \text{Black 3-brane (AdS}\_5\times\text{S}^5)\]
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warm-up example:

**Pure Yang-Mills**
*(bosonic)*
Wilson's lattice gauge theory

\[ S = -\beta N \sum_{\vec{x}} \sum_{\mu \neq \nu} \text{Tr} \left( U_{\mu,\vec{x}} U_{\nu,\vec{x}+\hat{\mu}} U_{\mu,\vec{x}+\hat{\nu}} U_{\nu,\vec{x}} \right) \]

Unitary link variable

\[ U_{\mu,\vec{x}} = e^{i\alpha A_{\mu}(x)} \]

\( \alpha \) : lattice spacing

\[ \beta = 1/(g_{YM}^2(\alpha) \cdot N) \]

\[ S = \frac{1}{4g_{YM}^2} \int d^4x \text{Tr} F_{\mu\nu}^2 + O(a^4) \]
‘Exact’ symmetries

• Gauge symmetry

\[ U_{\mu,\vec{x}} \rightarrow \Omega(x) U_{\mu,\vec{x}} \Omega(x + \hat{\mu})^\dagger \]

• 90 degree rotation

• discrete translation

• Charge conjugation, parity

These symmetries exist *at discretized level*. 
Continuum limit \( a \to 0 \) respects exact symmetries at discretized level.

What happens if the gauge symmetry is explicitly (not spontaneously) broken, (e.g. the sharp momentum cutoff prescription)?

Exact symmetries at discretized level

gauge invariance, translational invariance, rotationally invariant, ... in the continuum limit.
• We are interested in low-energy, long-distance physics (compared to the lattice spacing $a$).

• So let us integrate out high frequency modes.

Then...

gauge symmetry breaking radiative corrections can appear.

To kill them, one has to add counterterms to lattice action, whose coefficients must be fine-tuned!

‘fine tuning problem’

This is the reason why we must preserve symmetries exactly.
Super Yang-Mills
‘No-Go’ for lattice SYM

- SUSY algebra contains infinitesimal translation.
  \[ \{ Q, \bar{Q} \} \sim \partial \]

- Infinitesimal translation is broken on lattice by construction.

- So it is impossible to keep all supercharges exactly on lattice.

- Still it is possible to preserve a part of supercharges. (subalgebra which does not contain \( \partial \))
Strategy

Use other exact symmetries and/or a few exact SUSY to forbid SUSY breaking radiative correction.

• 1d : no problem thanks to UV finiteness. Lattice is not needed; momentum cutoff method is much more powerful. (M.H.-Nishimura-Takeuchi 2007)

• 2d : lattice with a few exact SUSY+R-symmetry
  • works even nonperturbatively (←simulation) (Kanamori-Suzuki 2008, M.H.-Kanamori 2009, 2010)
• 3d N=8 : “Hybrid” formulation:
  BMN matrix model + fuzzy sphere
  (Maldacena-Seikh Jabbari-Van Raamsdonk 2002)

• 4d N=1 pure SYM: lattice chiral fermion assures SUSY
  (Kaplan 1984, Curci-Veneziano 1986)

• 4d N=4:
  • again “Hybrid” formulation: Lattice + fuzzy sphere
    (M.H.-Matsuura-Sugino 2010, M.H. 2010)
  • Large-N Eguchi-Kawai reduction (Ishii-Ishiki-Shimasaki-Tsuchiya, 2008)
  • Another Matrix model approach (Heckmann-Verlinde, 2011)
  • recent analysis of 4d lattice:
    Fine tuning is needed, but only for 3 bare lattice couplings.
    (Catterall-Dzienkowski-Giedt-Joseph-Wells, 2011)
SIGN PROBLEM
Fermions appear in a bilinear form. 
(if not.. make them bilinear by introducing auxiliary fields!)

\[ S = S_B + S_F, \quad S_F = \int d^4x \bar{\psi} D\psi \]

\[ D = \gamma^\mu (\partial_\mu - iA_\mu) \]

Fermions appear in a bilinear form. 
(can be integrated out by hand.

\[ \int [dA][d\psi] e^{-S_B[A] - S_F[A,\psi]} = \int [dA] \det D[A] \cdot e^{-S_B[A]} \]

So, simply use the ‘effective action’,

\[ S_{eff}[A] = S_B[A] - \log \det D[A] \]

(crucial assumption : \( \det D > 0 \))
Sign problem

- ‘Probability’ must be real positive.
- Life is sometimes hard... path integral weight $e^{-S}$ can be \textit{complex}! (after the Wick rotation)
  - Chern-Simons term (pure imaginary!)
  - Finite baryon chemical potential
  - Yukawa coupling
  - Super Yang-Mills

Such path integral measures cannot be generated by the Markov-chain Monte-Carlo method :'-(
‘reweighting method’

• Use the ‘phase-quenched’ effective action

\[ S_{eff}[A] = S_B[A] - \log | \det D[A]| \]

• Phase can be taken into account by the ‘phase reweighting’:

\[
\langle \mathcal{O} \rangle = \frac{\int [dA] \det D \cdot e^{-S_B} \cdot \mathcal{O}}{\int [dA] \det D \cdot e^{-S_B}} \\
= \frac{\int [dA](\text{phase}) \cdot | \det D| \cdot e^{-S_B} \cdot \mathcal{O}/ \int [dA]| \det D| \cdot e^{-S_B}}{\int [dA](\text{phase}) \cdot | \det D| \cdot e^{-S_B}/ \int [dA]| \det D| \cdot e^{-S_B}} \\
= \frac{\langle (\text{phase}) \cdot \mathcal{O} \rangle_{\text{phase quench}}}{\langle (\text{phase}) \rangle_{\text{phase quench}}} 
\]
usually the reweighting does not work in practice...

• violent phase fluctuation
  → both numerator and denominator becomes almost zero. \(0/0 = ??\)

• vacua of full and phase-quenched model can disagree.
  ‘overlapping problem’
Miracles happen in SYM!

• Almost no phase except for very low temperature and/or SU(2).

• Even when the phase fluctuates, phase quench gives right answer. (‘right’ in the sense it reproduces gravity prediction.)

• Can be justified numerically.
  (M.H.-Nishimura-Sekino-Yoneya 2011)

This is the only theory in which we can believe in any miracle.

(D.B. Kaplan 2010, private communication.)
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(6) ABJM Theory and M-theory ('membrane mini revolution')
\[ S = \frac{N}{\lambda} \int dt \ Tr \left\{ \frac{1}{2} (D_t X_i)^2 - \frac{1}{4} [X_i, X_j]^2 \right. \\
\left. + \frac{1}{2} \bar{\psi} D_t \psi - \frac{1}{2} \bar{\psi} \gamma^i [X_i, \psi] \right\} \]

- Dimensional reduction of 4d N=4 (or 10d N=1)
- D0-brane effective action
- Matrix model of M-theory
- **gauge/gravity duality** → dual to black 0-brane

Simple but can be more interesting than AdS$_5$/CFT$_4$ from string theory point of view!
• Matrix quantum mechanics is **UV finite**.

  *No fine tuning!*

  (4d N=4 is also UV finite, but that relies on cancellations of the divergences...)

• **We don’t have to use lattice. Just fix the gauge & introduce momentum cutoff!**
  (M.H.-Nishimura-Takeuchi, 2007)
• Take the static diagonal gauge

\[ A_0(t) = \text{diag}(\alpha_1, \cdots, \alpha_N)/\beta \]
\[ \alpha_1, \cdots, \alpha_N \in (-\pi, \pi] \]

• Add Faddeev-Popov term

\[ S_{FP} = -\sum_{a \neq b} \log \left| \sin \frac{\alpha_a - \alpha_b}{2} \right| \]

• Introduce momentum cutoff \( \Lambda \)

\[ X_i(t) = \sum_{n=-\Lambda}^{\Lambda} \tilde{X}_i(n) e^{2\pi i n t/\beta} \]
Gravity side
Gauge/gravity duality conjecture
(Maldacena 1997; Itzhaki-Maldacena-Sonnenschein-Yankielowicz 1998)

“(p+1)-d maximally supersymmetric U(N) YM and type II superstring on black p-brane background are equivalent”

\[ p=3 : \text{AdS}_5/\text{CFT}_4 \]
\[ p<3 : \text{nonAdS}/\text{nonCFT} \]

large-N, strong coupling = SUGRA
finite coupling = \( \alpha' \) correction
finite \( N = g_s \) correction
black p-brane solution
(Horowitz-Strominger 1991)

\[ ds^2 = \alpha' \left\{ \frac{U^{7-p}}{g_{YM} \sqrt{d_p N}} \left[ - \left( 1 - \frac{U_0^{7-p}}{U^{7-p}} \right) dt^2 + \sum_{i=1}^{p} dy_i^2 \right] \right. \]
\[
+ \frac{g_{YM} \sqrt{d_p N}}{U^{7-p} \left( 1 - \frac{U_0^{7-p}}{U^{7-p}} \right)} dU^2 \left. \right. \]
\[
+ g_{YM} \sqrt{d_p N U^{p-3}} d\Omega_{8-p}^2 \right\},
\]

\[ e^\phi = (2\pi)^{2-p} g_{YM}^2 \left( \frac{g_{YM}^2 d_p N}{U^{7-p}} \right)^{\frac{3-p}{4}} , \]

\[ d_p = 2^{7-2p} \pi^{\frac{9-3p}{2}} \Gamma \left( \frac{7-p}{2} \right) , \]

SUGRA is valid at
\[ \lambda^{1/3} N^{-4/21} \ll U \ll \lambda^{1/3} \quad (p = 0) \]
Difference from AdS/CFT

• When $p<3$, 't Hooft coupling $\lambda$ is *dimensionful*. It sets the length scale of the theory.

• 't Hooft coupling can be set $\lambda=1$, by rescaling fields and coordinate.

\[
T_{D0} = \frac{7}{4\pi \sqrt{d_0 \lambda}} U_0^{5/2}
\]

Hawking temperature

‘strong coupling’

= low temperature

$\lambda^{-1/3} T \ll 1$. 
The dictionary

Gravity

- ADM mass
- minimal surface
- mass of field excitation

SYM

- Energy density
- Wilson/Polyakov loop
- scaling dimension
“moduli” problem

• There is a flat direction even at quantum level.
  
  \[ [X_i, X_j] = 0 \]

• In 1d (and 2d), it is not a “moduli space”; value of the configuration should be determined dynamically.

• SUGRA (\(\mathcal{N}=\infty\)) suggests the black zero-brane is stable.
  \[ X_1 \simeq X_2 \simeq \cdots \simeq X_9 \simeq 0 \]

• 1/N correction (\(g_s\) correction) should give an instability: Hawking radiation.

• Instability should disappear at large-N and/or at high temperature. And it does happen in simulations!
Numerical observation

• Because we are interested in the black hole, we take the initial configuration to be

\[ X_1 = X_2 = \cdots = X_9 = 0 \]

• The bound state of eigenvalues is (meta-)stable at large enough \( N \) and high enough \( T \).

• The bound state appears again at low-temperature \( (T<0.2 \text{ for } SU(2)) \).

\( \leftarrow \) attraction due to fermion zero-mode.

(Aoki-Iso-Kawai-Kitazawa-Tada 1998)

More stable with pbc.
ADM mass vs energy density

$$E_{D0} = \frac{9}{2^{11} \pi^{\frac{13}{2}} \Gamma\left(\frac{9}{2}\right) \lambda^2} N^2 U_0^7$$

$$\frac{1}{N^2} E_{D0} \sim 7.4 \ T^{2.8} \quad (\lambda = 1)$$

at large-N & low temperature (strong coupling)
Anagnostopoulos-M.H.-Nishimura-Takeuchi 2007,
M.H.-Hyakutake-Nishimura-Takeuchi 2008
α’ correction

• deviation from the strong coupling (low temperature) corresponds to the α’ correction (classical stringy effect).

• The α' correction to SUGRA starts from (α')³ order

• Correction to the BH mass :
  (α'/R²)³ ∼ T¹.8

• $E/N^2=7.41T^{2.8} - 5.58T^{4.6}$

‘prediction’ by SYM simulation
M.H.-Hyakutake-Nishimura-Takeuchi 2008

\[ SUGRA \]

\[ SUGRA + \alpha' \]
slope=4.6

finite cutoff effect

higher order correction

M.H.-Hyakutake-Nishimura-Takeuchi 2008
4-SUSY MQM
(M.H.-Matsuura-Nishimura-Robles 2010)

Exponential $\frac{E}{N^2} \sim \exp(-a/T)$ rather than power
$\rightarrow$ consistent with the absence of the zero-energy normalizable state
Polyakov loop with scalar

(Maldacena 1998; Rey-Yee 1998)

\[ W = TrP \exp \left( \int (iA + X) dt \right) \]

\[ \log\langle W \rangle \sim \langle \log W \rangle \sim \text{area of minimal surface} \]

boundary=Polyakov loop
1.89/T^{0.6} \sim \text{area}

M.H.-Miwa-Nishimura-Takeuchi, 2008
Correlation functions 
(GKPW relation)

• AdS/CFT (D3-brane) → GKPW relation 
  (Gubser-Klebanov-Polyakov 1998, Witten 1998)

• Similar relation in D0-brane theory :

  “generalized” conformal dimension
  $\leftrightarrow$ mass of field excitations
  (Sekino-Yoneya 1999)

$$\langle \mathcal{O}(t)\mathcal{O}(0) \rangle \sim t^{-p}$$
  calculable via SUGRA
two-point functions, SU(3), pbc

\[ J_{l; i_1 \ldots i_l}^{+i_j} \equiv \frac{1}{N} \cdot Str \left( F_{ij} X_{i_1} \cdots X_{i_l} \right) \quad (F_{ij} \equiv -i [X_i, X_j]) \]

(M.H.-Nishimuea-Sekino-Yoneya 2009)
SU(3) is large-N :) 

two-point functions, SU(3), pbc

\[ J_{l; i_1 \cdots i_l}^{+ij} \equiv \frac{1}{N} \cdot Str \left( F_{ij} X_{i_1} \cdots X_{i_l} \right) \quad (F_{ij} \equiv -i[X_i, X_j]) \]

(M.H.-Nishimuea-Sekino-Yoneya 2009)
two-point functions, SU(2), pbc

(M.H.-Nishimura-Sekino-Yoneya 2011)
SU(2) is large-N

two-point functions, SU(2), pbc

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Basic ideas

• Keep a few supercharges exact on lattice.

• Use it (and other discrete symmetries) to forbid SUSY breaking radiative corrections. (Kaplan-Katz-Unsal 2002)

• Only “extended” SUSY can be realized for a technical reason. (4, 8 and 16 SUSY)

• Below we consider 16 SUSY theory.
Several lattice theories exists (from around 2002-2005)

- Cohen, Kaplan, Katz, Unsal
- Sugino
- Catterall
- Suzuki, Taniguchi
- D’Adda, Kanamori, Kawamoto, Nagata

Explained below (conceptually the simplest, according to my taste)
\[ S_0 = \frac{1}{g_{2d}^2} \int d^2 x \ Tr \left\{ F_{12}^2 + (D_\mu X^I)^2 - \frac{1}{2} [X^I, X^J]^2 \right. \\
+ \Psi^T (D_1 + \gamma_2 D_2) \Psi + i \Psi^T \gamma_I [X^I, \Psi] \left. \right\} \]

**Q-exact form**

\[ S_0 = Q^{(0)}_+ Q^{(0)}_- F^{(0)}, \]

\[ F^{(0)} = \frac{1}{g_{2d}^2} \int d^2 x \ Tr \left\{ -i B_A \Phi_A - \frac{1}{3} \epsilon_{ABC} B_A [B_B, B_C] \right. \\
- \psi_{+\mu} \psi_{-\mu} - \rho_{+\mu} \rho_{-\mu} - \chi_{+A} \chi_{-A} - \frac{1}{4} \eta_{+} \eta_{-} \left. \right\}, \]

\[ \Phi_1 = 2 (-D_1 X_3 - D_2 X_4), \quad \Phi_2 = 2 (-D_1 X_4 + D_2 X_3), \]
\[ \Phi_3 = 2 (-F_{12} + i [X_3, X_4]). \]
\[ Q^{(0)}_{\pm} A_\mu = \psi_{\pm\mu}, \quad Q_{\pm} \psi_{\pm\mu} = \pm i D_\mu \phi_{\pm}, \]
\[ Q^{(0)}_{\pm} \psi_{\pm\mu} = \frac{i}{2} D_\mu C \mp \tilde{H}_\mu, \]
\[ Q^{(0)}_{\pm} \tilde{H}_\mu = [\phi_{\pm}, \psi_{\mp\mu}] \mp \frac{1}{2} [C, \psi_{\pm\mu}] \mp \frac{i}{2} D_\mu \eta_{\pm}, \]
\[ Q^{(0)}_{\pm} X_i = \rho_{\pm i}, \quad Q^{(0)}_{\pm} \rho_{\pm i} = \mp [X_i, \phi_{\pm}], \]
\[ Q^{(0)}_{\mp} \rho_{\pm i} = -\frac{1}{2} [X_i, C] \mp \tilde{h}_i, \]
\[ Q^{(0)}_{\pm} \tilde{h}_i = [\phi_{\pm}, \rho_{\mp i}] \mp \frac{1}{2} [C, \rho_{\pm i}] \pm \frac{1}{2} [X_i, \eta_{\pm}], \]
\[ Q^{(0)}_{\pm} B_A = \chi_{\pm A}, \quad Q^{(0)}_{\pm} \chi_{\pm A} = \pm [\phi_{\pm}, B_A], \]
\[ Q^{(0)}_{\mp} \chi_{\pm A} = -\frac{1}{2} [B_A, C] \mp H_A, \]
\[ Q^{(0)}_{\pm} H_A = [\phi_{\pm}, \chi_{\mp A}] \pm \frac{1}{2} [B_A, \eta_{\pm}] \mp \frac{1}{2} [C, \chi_{\pm A}], \]
\[ Q^{(0)}_{\pm} C = \eta_{\pm}, \quad Q^{(0)}_{\pm} \eta_{\pm} = \pm [\phi_{\pm}, C], \]
\[ Q^{(0)}_{\mp} \eta_{\pm} = \mp [\phi_{\mp}, \phi_{\mp}], \]
\[ Q^{(0)}_{\pm} \phi_{\pm} = 0, \quad Q^{(0)}_{\mp} \phi_{\pm} = \mp \eta_{\pm}. \]
Nilpotency

\[ (Q_+^{(0)})^2 = (Q_-^{(0)})^2 = \{Q_+^{(0)}, Q_-^{(0)}\} = 0 \]

\[ \Rightarrow Q_\pm^{(0)} S^{(0)} = 0 \] can be seen manifestly.

Strategy

Realize this SUSY algebra on lattice. Then the lattice action has two exact SUSY and SU(2)_R.

But how?

... trial and error!
\[ Q_{\pm} U_{\mu}(x) = i \psi_{\pm \mu}(x) U_{\mu}(x), \]
\[ Q_{\pm} \psi_{\pm \mu}(x) = i \psi_{\pm \mu}(x) \psi_{\pm \mu}(x) \pm i D_{\mu} \phi_{\pm}(x), \]
\[ Q_{\mp} \psi_{\pm \mu}(x) = \frac{i}{2} \{ \psi_{+ \mu}(x), \psi_{- \mu}(x) \} + \frac{i}{2} D_{\mu} C(x) \mp \tilde{H}_{\mu}(x), \]
\[ Q_{\pm} \tilde{H}_{\mu}(x) = -\frac{1}{2} \left[ \psi_{\mp \mu}(x), \phi_{\pm}(x) + U_{\mu}(x) \phi_{\pm}(x + \hat{\mu}) U_{\mu}(x)^\dagger \right] \]
\[ \pm \frac{1}{4} \left[ \psi_{\pm \mu}(x), C(x) + U_{\mu}(x) C(x + \hat{\mu}) U_{\mu}(x)^\dagger \right] \]
\[ \mp \frac{i}{2} D_{\mu} \eta_{\pm}(x) \pm \frac{1}{4} \left[ \psi_{\pm \mu}(x) \psi_{\pm \mu}(x), \psi_{\mp \mu}(x) \right] \]
\[ \mp \frac{i}{2} \left[ \psi_{\pm \mu}(x), \tilde{H}_{\mu}(x) \right] \]

\[ D_{\mu} A(x) \equiv U_{\mu}(x) A(x + \hat{\mu}) U_{\mu}(x)^\dagger - A(x) \]

Sugino, 2003
\[ \Phi_1(x) = 2 \left( -D_1 X_3(x) - D_2 X_4(x) \right), \]
\[ \Phi_2(x) = 2 \left( -D_1^* X_4(x) + D_2^* X_3(x) \right), \]
\[ \Phi_3(x) = \frac{i(U_{12}(x) - U_{21}(x))}{1 - \epsilon^{-2} \|1 - U_{12}(x)\|^2} + 2i[X_3(x), X_4(x)] \]

↑ admissibility condition

• Gauge part is not \( \text{Tr}(\text{plaquette}) \) but \( \text{Tr}(\text{plaquette})^2 \)

• To exclude wrong vacua with plaquette = -1, we employ the admissibility.

• The action is set to be zero unless
\[ \|1 - U_{12}(x)\| < \epsilon \text{ for } \forall x \]

Sugino, 2003
Absence of fine tuning
(to all order in perturbation)
(Cohen-)Kaplan-Katz-Unsal, 2002&2003

• Possible correction from UV is

\[
\left( \frac{1}{g_{2d}^2} c_0 a^{p-4} + c_1 a^{p-2} + g_{2d}^2 c_2 a^p + \cdots \right) \int d^2 x \, O_p(x)
\]

tree

up to \( \log(a) \), where

\[
O_p(x) = \varphi(x) \partial^\beta \psi(x)^{2\gamma}, \quad p = \alpha + \beta + 3\gamma
\]

• Only \( p=1,2 \) are dangerous.

\[
\varphi, \varphi^2
\]

(\( \partial \varphi \) is a total derivative)

\[SU(2)_R\] allows only \( \text{TrB}_A \) and \( \text{TrX}_i \).

Exact SUSY kills them.

\( \varphi^2 \) term is forbidden in a similar manner.
Does it work at nonperturbative level?
4 SUSY model (dimensional reduction of 4d N=1; sign-free) has been studied extensively.

- Comparison with analytic results at small volume & large-N behaviors. (M.H.-Kanamori 2009)
- Comparison to Cohen-Kaplan-Katz-Unsal model. (M.H.-Kanamori 2010)

All results supports the emergence of the correct continuum limit without fine tuning.
Polyakov loop vs compactification radius
SU(2), periodic b.c.  (M.H.-Kanamori 2010)
$\sqrt{Tr(X_i(x))^2/N}$ vs compactification radius
SU(2), periodic b.c.  (M.H.-Kanamori 2010)
Application: black hole/black string transition

Susskind, Barbon-Kogan-Rabinovici, Li-Martinec-Sahakian, Aharony-Marsano-Minwalla-Wiseman,…

SYM simulation: Catterall-Wiseman, 2010
• Consider 2d U(N) SYM on a spatial circle. It describes N D1-branes in $\mathbb{R}^{1,8} \times S^1$, winding on $S^1$.
• T-dual picture : N D0-branes in $\mathbb{R}^{1,8} \times S^1$.
• Wilson line phase = position of D0

uniform distribution = ‘black string’
localized distribution = ‘black hole’
Fix the mass (or temperature) and shrink the compactification radius. Then...

black hole

nonuniform black string

uniform black string
Counterpart in SYM

$= \text{center symmetry breakdown}$

- Wilson line phase = position of D0
  \[ W = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_N}) \]
- Center symmetry
  \[ \theta_i \rightarrow \theta_i + \text{const.} \]

Uniform = center unbroken
\[ \left< \frac{1}{N} TrW \right> = 0 \]

Non-uniform = center broken
\[ \left< \frac{1}{N} TrW \right> \neq 0 \]
Phase diagram
(Theoretical prediction)

(Temperature)$^{-1}$

Low temperature:
1st order
BH $\rightarrow$ uniform BS
(Aharony et al, 2004)

High temperature:
2nd + 3rd
BH $\rightarrow$ nonuniform BS
$\rightarrow$ uniform BS
(Kawahara et al, 2007)

Figure from Catterall-Wiseman, 2010
Value of spatial Wilson loop ('t Hooft loop)

SU(3)  

SU(4)
SU(4) gives bigger value of 't Hooft loop than SU(3)

't Hooft loop = 0.5

~ BH/BS transition
Plan

(1) Gauge/Gravity duality (AdS/CFT) and Super Yang-Mills

(2) Why lattice SUSY is hard (fine tuning, sign problem)

(3) How to put SYM on computer. (i).
   D0-brane quantum mechanics

(4) How to put SYM on computer. (ii).
   (1+1)-d theories

(5) How to put SYM on computer. (iii).
   (2+1)-d and (3+1)-d theories

(6) ABJM Theory and M-theory (‘membrane mini revolution’)
• 4d N=1 pure SYM: lattice chiral fermion assures SUSY (Kaplan 1984, Curci-Veneziano 1986)

• 4d N=4: lattice formulation (Kaplan-Unsal, Sugino, Catterall, ...) ⇒ fine tunings remain.

Fine tuning is needed, but only for 3 bare lattice couplings, at least perturbatively.
(Catterall-Dzienkowski-Giedt-Joseph-Wells, 2011) ⇒ doable?

• 4d N=4: “Hybrid” formulation:
  Lattice + fuzzy sphere

  (M.H.-Matsuura-Sugino 2010, M.H. 2010)

• Large-N Eguchi-Kawai reduction (Ishii-Ishiki-Shimasaki-Tsuchiya, 2008)

• Another Matrix model approach (Heckmann-Verlinde, 2011)
Outline

• Main idea
• lattice formulation of ‘2d BMN’
• uplift to 4d
Fuzzy sphere formulation of 3d maximal SYM
(Maldacena-Sheikh Jabbari-van Raamsdonk, 2003)

• Start with the Berenstein-Maldacena-Nastase Matrix model, which can be formulated without fine tuning.

\[ S = \int dt \ Tr \left( \frac{1}{2} (D_t X_I)^2 - \frac{1}{4} [X_I, X_J]^2 + \frac{i \mu}{3} \epsilon^{abc} X_a X_b X_c + \frac{\mu^2}{18} X_a^2 + \frac{\mu^2}{72} X_i^2 \right) \]

I,J=1,...,9; a,b,c=1,2,3; i=4,...,9

• BMN model has (modified) 16 SUSY
Fuzzy sphere solution

\[-[X_b, [X_a, X_b]] + i \mu \epsilon^{abc} X_b X_c + \frac{\mu^2}{9} X_a = 0\]

\[\rightarrow X_a = \frac{\mu}{3} J_a, \quad [J_a, J_b] = i \epsilon_{abc} J_c\]

- preserves 16 SUSY. Around it one obtains (1+2)-d SYM on noncommutative space.

- D2 out of D0 (Myers effect)

- With maximal SUSY, commutative limit of the noncommutative space is smooth.

(no UV/IR mixing) (Matusis-Susskind-Toumbas ’00)
Q. How can we construct 4d theory (D3-brane)?

A. From D1-branes through the Myers effect.

**Crucial point**

*D1-brane theory (2d SYM) can be formulated on lattice without fine tuning!*

- Take 2d continuum limit first, then large-N
- Similar anisotropic continuum limit was taken on 4 lattice, in order to reduce the number of fine tuning parameters. (Kaplan-Katz-Unsal, 2003)
- Analogous to “deconstruction” of 5d out of 4d (Arkani Hamed-Cohen-Georgi, 2001)
Outline

• Main idea
• lattice formulation of ‘2d BMN’
• uplift to 4d
\[
\Delta S = \frac{1}{g_{2d}^2} \int d^2 x \text{ Tr} \left\{ \frac{2M^2}{81} \left( B_A^2 + X_i^2 \right) \right. \\
\left. - \frac{M}{2} C[\phi_+, \phi_-] + \frac{M^2}{9} \left( \frac{C^2}{4} + \phi_+ \phi_- \right) \right. \\
\left. + \frac{2M}{3} \psi_{+\mu} \psi_{-\mu} + \frac{2M}{9} \rho_{+i} \rho_{-i} + \frac{4M}{9} \chi_{+A} \chi_{-A} \right. \\
\left. - \frac{M}{6} \eta_+ \eta_- - \frac{4iM}{9} B_3 \left( F_{12} + i[X_3, X_4] \right) \right\}
\]

\[
\Delta Q_{\pm} \tilde{H}_\mu = \frac{M}{3} \psi_{\pm\mu}, \quad \Delta Q_{\pm} \tilde{h}_i = \frac{M}{3} \rho_{\pm i}, \\
\Delta Q_{\pm} H_A = \frac{M}{3} \chi_{\pm A}, \quad \Delta Q_{\pm} \eta_\pm = \frac{2M}{3} \phi_\pm, \\
\Delta Q_{\mp} \eta_\pm = \pm \frac{M}{3} C.
\]

M.H.-Matsuura-Sugino, 2010
\[ Q_+^2 = \frac{M}{3} J_{++}, \quad Q_-^2 = -\frac{M}{3} J_{--}, \]
\[ \{Q_+, Q_-\} = -\frac{M}{3} J_0, \]

J : SU(2)_R generator;
fermions with +/- form doublets

\[ S = \left( Q_+ Q_- - \frac{M}{3} \right) F \]

Q-closed!
Absence of fine tuning
(to all order in perturbation)

(April the same as (Cohen-)Kaplan-Katz-Unsal, 2002 & 2003)

- Possible correction from UV is
  \[
  \left( \frac{1}{g_{2d}^2} c_0 a^{p-4} + c_1 a^{p-2} + g_{2d}^2 c_2 a^p + \cdots \right) \int d^2 x \, O_p(x)
  \]
  up to \( \log(aM) \), where
  \[
  O_p(x) = M^m \varphi(x)^\alpha \partial^\beta \psi(x)^{2\gamma}, \quad p \equiv m + \alpha + \beta + 3\gamma
  \]

- Only \( p=1,2 \) are dangerous.
  \[
  \varphi, \quad M \varphi, \quad \varphi^2
  \]
  (\( \partial \varphi \) is a total derivative)

\( SU(2)_R \) allows only \( \text{Tr} B_A \) and \( \text{Tr} X_i \).

Exact SUSY kills them.

\( \varphi^2 \) term is forbidden in a similar manner.
Outline

- Main idea
- lattice formulation of ‘2d BMN’
- uplift to 4d
• Firstly, take a continuum limit as 2d theory.
• Secondly, take a $k$-coincident fuzzy sphere solution. Then U($k$) SYM on fuzzy sphere is obtained.

\[ L_a = L_a^{(n)} \otimes 1_k \quad N = k(2n+1) \]

noncommutativity : $\theta \sim 1/(M^2 n)$

UV/IR momentum cutoff : $M_n, M$

Coupling constant : $g_{4d}^2 = 4\pi \theta g_{2d}^2$
• Take a flat noncommutative space limit, i.e. $N \to \infty$ fixing $\Theta$

• Because 14 of 16 SUSYs are softly broken, additional “UV divergence” seems to appear.

$$M^p (\log N)^q$$

• But now $M$ goes to zero as $M \sim 1/\sqrt{N}$ and hence there is no “UV divergence”. So SYM on flat noncommutative space is obtained.

• In the end we take the commutative limit.
• 4d N=4 can be formulated without requiring parameter fine tuning, at least to all order in perturbation.

• UV finiteness is the key to justify the use of fuzzy sphere. Other UV finite theories may be formulated in a similar manner.

• Simulation? -- hopefully in near future! (2d lattice code is ready. (Buchoff-M.H.-Matsuura))
Summary of SYM part
maximal SYM can be put on computer, by combining lattice and non-lattice methods.

Sign problem? No problem!

1d (non-lattice) : nice & precise results.

2d (lattice) : ongoing.

3d, 4d (fuzzy sphere, lattice) : coming soon.

For other theories (e.g. SUSY QCD) new ideas are needed.

Does gauge/gravity duality hold at finite N?

‘simulation of quantum superstring’ is within reach.
Plan

(1) Gauge/Gravity duality (AdS/CFT) and Super Yang-Mills

(2) Why lattice SUSY is hard (fine tuning, sign problem)

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    D0-brane quantum mechanics

(4) How to put SYM on computer. (ii).
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(6) ABJM Theory and M-theory (‘membrane mini revolution’)
M-theory

- Strong coupling limit of type IIA superstring.
- Low-energy effective theory is 11d supergravity.
- Theory of membrane.
- string = membrane winding on 11-th dimension.
AdS/CFT corresponding can hold also in M-theory.

D3-brane in IIB string $\rightarrow$ AdS$_5 \times$ S$^5$

M5-brane in M-theory $\rightarrow$ AdS$_7 \times$ S$^4$

M2-brane in M-theory $\rightarrow$ AdS$_4 \times$ S$^7$

D2-brane $\rightarrow$ M2-brane

2d maximal SYM $\rightarrow$ 3d CFT

CFT, maximal SUSY, correct moduli space, ...
ABJM theory
(Aharony-Bergman-Jafferis-Maldacena, 2008)

\[ k \, \text{Tr} \left\{ \frac{e^{\mu \nu \rho}}{2} \left( -A_\mu \partial_\nu A_\rho - \frac{2}{3} A_\mu A_\nu A_\rho + \tilde{A}_\mu \partial_\nu \tilde{A}_\rho + \frac{2}{3} \tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\rho \right) \right. \\
\left. + (-D_\mu \tilde{\Phi}^\alpha D^\mu \Phi_\alpha + i \tilde{\Psi}^\alpha \bar{D} \Psi_\alpha) - i \epsilon^{\alpha \beta \gamma \delta} \Phi_\alpha \bar{\Psi}_\beta \Phi_\gamma \bar{\Psi}_\delta + i \epsilon_{\alpha \beta \gamma \delta} \bar{\Phi}^\alpha \Psi^\beta \bar{\Phi}^\gamma \Psi_\delta \\
+ i \left( - \bar{\Psi}_\beta \Phi_\alpha \tilde{\Phi}^\alpha \Psi^\beta + \Psi_\beta \bar{\Phi}^\alpha \Phi_\alpha \bar{\Psi}^\beta + 2 \bar{\Psi}_\alpha \Phi_\beta \tilde{\Phi}^\alpha \Psi^\beta - 2 \Psi^\beta \bar{\Phi}^\alpha \Phi_\beta \bar{\Psi}_\alpha \right) \\
+ \frac{1}{3} \left( \Phi_\alpha \Phi_\beta \Phi_\gamma \Phi_\delta \bar{\Phi}^\alpha \bar{\Phi}^\beta \bar{\Phi}^\gamma \bar{\Phi}^\delta + \Phi_\alpha \bar{\Phi}^\alpha \Phi_\beta \Phi_\gamma \Phi_\delta \bar{\Phi}^\beta \bar{\Phi}^\gamma \bar{\Phi}^\delta + 4 \Phi_\beta \bar{\Phi}^\alpha \Phi_\gamma \Phi_\delta \bar{\Phi}^\beta \bar{\Phi}^\gamma \bar{\Phi}^\delta \Phi_\alpha \bar{\Phi}^\alpha - 6 \Phi_\gamma \bar{\Phi}^\gamma \Phi_\beta \bar{\Phi}^\beta \Phi_\alpha \bar{\Phi}^\alpha \Phi_\delta \bar{\Phi}^\delta \right) \right. \]

(developed out of earlier works by Schwarz, Bagger-Lambert, etc etc...)

A  
B  
J  
M
ABJM theory
(Aharony-Bergman-Jafferis-Maldacena, 2008)

- 3d $U(N)_k \times U(N)_{-k}$ Superconformal Chern-Simons-Matter theory with $N=6$ SUSY ($N=8$ when $k=1,2$)
- Correct moduli ($N$ M2-branes on $\mathbb{R}^{1,2} \times \mathbb{R}^8 / \mathbb{Z}_k$)
- the IR fixed point of 3d maximal super Yang-Mills
- dual to IIA string on $AdS_4 \times \mathbb{CP}^3$ in the 't Hooft large-$N$ limit ($\lambda = N/k$ fixed)
- dual to M-theory (or 11d SUGRA) on $AdS_4 \times S^7 / \mathbb{Z}_k$ at large-$N$, $k^5 \ll N$
\( \lambda = \frac{N}{k} \)

- Perturbative gauge theory
- Tree-level string (\( \alpha' \) correction)
- Quantum string
- M-theory
- IIA SUGRA
M-theory geometry

N M2-branes on $R^{1,2} \times R^8 / \mathbb{Z}_k$

near horizon geometry $= \text{AdS}_4 \times S^7 / \mathbb{Z}_k$

$$ds^2 = \frac{R^2}{4} ds^2_{\text{AdS}_4} + R^2 ds^2_{S^7}$$

$$R / l_p = (2^5 \pi^2 N k)^{1/6}$$

classical description fails when $S^7 / \mathbb{Z}_k$ is smaller than the Planck scale.

$$\left( \frac{R}{l_p} \right) / k \sim N^{1/6} k^{-5/6} \gg 1$$ is needed.

$N \gg k^5$
Prediction from gravity side

- Free energy in IIA string region ($N/k$ fixed)

$$F = \frac{\pi \sqrt{2}}{3} \frac{N^2}{\sqrt{\lambda}}$$

- Free energy in M-theory region ($N \gg k^5$)

$$F = \frac{\pi \sqrt{2}}{3} \sqrt{kN^{3/2}}$$
So what’s new?

• It may be the definition of the M-theory!
  but...

M-theory region is too difficult, people will just copy what they did in AdS$_5$/CFT$_4$ to type IIA region...

(Ofer Aharony, journal club @Weizmann Institute, June 2008)

(1) analytically very difficult because M-theory appears outside the ‘t Hooft limit
(2) Simulation is also difficult
   (fine tuning, sign problem)

then... nothing new??
Quantum correction in gravity side can be estimated to some extent. (Bergman-Hirano, 2009)

Some quantities in the gauge theory (free energy, Wilson loop, etc) can be calculated by using the ‘localization’ at finite $N$. (Drukker-Marino-Putrov 2009-2011, Fuji-Hirano-Moriyama 2011)

Test of the gauge/gravity duality at finite-$N$ and quantum string level is within reach!

I believe there is something really new.

(Shinji Hirano, private communication @Niels Bohr Institute, June 2008)
Goal of this talk: study whole parameter region from the gauge theory!
Outline

• What is the localization?
• Combine the localization with Monte Carlo → explicit numbers at finite N and finite k
• Free energy
• Wilson loop
• Future directions
localization
supersymmetric Wilson loop in 4d N=4 SYM

\[ W = \frac{1}{N} Tr \ P \exp \left( \int ds \ (iA_\mu \dot{x}^\mu(s) + X_i \theta^i |\dot{x}(s)|) \right) \]

(\( \theta^i \) : constant unit vector)

circular loop \( \rightarrow \) half BPS

gravity side:

\[ \log <W> \sim \text{area of the minimal worldsheet} \]

boundary = Wilson loop
Drukker-Gross conjecture
(2004 Nobel Prize in physics)

- Circular Wilson loop is BPS and calculation simplifies. (Erickson-Semenoff-Zarembo 2000)
- It can be calculated from the Gaussian matrix model. (Drukker-Gross 2000)

\[
\langle W_{\text{circle}} \rangle = \left\langle \frac{1}{N} \Tr \exp(M) \right\rangle = \frac{1}{Z} \int \mathcal{D}M \frac{1}{N} \Tr \exp(M) \exp \left( -\frac{2N}{\lambda} \Tr M^2 \right)
\]
proof of Drukker-Gross conjecture (Pestun 2007)

• Deform the theory, keeping the expectation value of the Wilson loop unchanged.

\[ S \rightarrow S + tQV \]

\[ QS = 0, \quad QW = 0 \]

\[
\frac{d}{dt} \int [dX] \mathcal{O} e^{-S-tQV} = - \int [dX] \mathcal{O} \cdot (QV) e^{-S-tQV} \\
= - \int [dX] Q (\mathcal{O} \cdot Ve^{-tQV}) e^{-S} \\
= Z_{t=0} \times \langle Q\text{-exact} \rangle = 0
\]

• The action ‘localizes’ to the Gaussian matrix model.
only one constant mode of scalar survives and the theory reduces to a Gaussian matrix model

\[ QV \sim F_{\mu\nu}^2 + [X_I, X_J]^2 + X_I^2 + \cdots \]

(\textit{the calculation is long and complicated!!})
localization in ABJM

• The partition function of ABJM on $S^3$ reduces to ‘ABJM matrix model’
  (Kapustin-Willett-Yaakov 2009)

$$Z_{ABJM} = \frac{1}{N!^2} \int \frac{d^N \mu}{(2\pi)^N} \frac{d^N \nu}{(2\pi)^N} \prod_{i<j} \frac{\left[ 2 \sinh \left( \frac{\mu_i - \mu_j}{2} \right) \right]^2 \left[ 2 \sinh \left( \frac{\nu_i - \nu_j}{2} \right) \right]^2}{\prod_{i,j} \left[ 2 \cosh \left( \frac{\mu_i - \nu_j}{2} \right) \right]^2} \exp \left[ \frac{ik}{4\pi} \sum_{i=1}^{N} (\mu_i^2 - \nu_i^2) \right]$$

• Perturbative part (in string language) can be studied analytically even at strong coupling.
  (Drukker-Marino-Putrov 2009-2011, Fuji-Hirano-Moriyama 2011)
all order in $1/N$ around planar limit (Fuji-Hirano-Moriyama 2011)

$$\log Z^{ABJM} = \log \left( 2\pi C_1 Ai \left( \left( \frac{\pi}{\sqrt{2}} \left( \frac{N}{\lambda} \right)^2 \lambda_{\text{ren}}^{3/2} \right)^{2/3} \right) \right)$$

$$C_1 = \frac{1}{\sqrt{2}} \left( \frac{2\pi}{k} \right)^{-1/3} \quad \lambda_{\text{ren}} = \lambda - \frac{1}{24} - \frac{\lambda^2}{3N^2}$$

$$Ai(x) = \frac{1}{\pi} \int_0^\infty dt \cos \left( \frac{t^3}{3} + xt \right) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}}$$

Rem

World-sheet instanton $e^{-2\pi \sqrt{\lambda}}$ is not included.
Supposed to corresponds to all order in string perturbation theory.
problems still remain...

• Is it really correct? (actually we will find corrections later)

• Does it hold away from IIA region? (Marino-Putrov 2011: fixed k, large-N is also OK)

• World-sheet instantons?

• Wilson loops?

more importantly for me...

*I cannot follow such a complicated calculation :(

So, a numerical method is welcomed!
localization + Monte Carlo
Monte Carlo simulation

• expectation value = average with a probability weight $e^{-s}/Z$

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \mathcal{O}[A^{(i)}] = \frac{1}{Z_{YM}} \int dA_\mu \mathcal{O}[A] e^{-S_{YM}[A]} \equiv \langle \mathcal{O} \rangle$$

• generate configurations $\{A_\mu^{(i)}\}$ with this probability.

Monte Carlo method

• ‘probability’ must be real positive.
ABJM matrix model (again)

- Just a $2N$-variable (path-)integral. Perhaps Monte Carlo works?

$$Z_{ABJM} = \frac{1}{N!^2} \int \frac{d^N \mu}{(2\pi)^N} \frac{d^N \nu}{(2\pi)^N} \prod_{i<j} \left[ 2 \sinh \left( \frac{\mu_i - \mu_j}{2} \right) \right]^2 \left[ 2 \sinh \left( \frac{\nu_i - \nu_j}{2} \right) \right]^2 \Pi_{i,j} \left[ 2 \cosh \left( \frac{\mu_i - \nu_j}{2} \right) \right]^2 \exp \left[ \frac{ik}{4\pi} \sum_{i=1}^{N} (\mu_i^2 - \nu_i^2) \right]$$

- Actually it’s impossible because of the sign problem. (Monte Carlo is applicable only when the action is real.)
Then change variables and get **sign-free action**!

$$Z_{ABJM} = \frac{1}{2^N N!} \int \frac{d^N x}{(2\pi k)^N} \frac{\prod_{i<j} \tanh^2 \left( \frac{x_i-x_j}{2k} \right)}{\prod_i 2 \cosh \left( \frac{x_i}{2} \right)}$$

**sign free!**

Now we can use Monte Carlo.
How to get sign-free action

- Similar to the S-duality.
- Technical, long and tedious, impossible to understand without following the calculation. So let’s *skip* the derivation.

**Key relations:**

**Cauchy identity**

\[
\frac{\prod_{i<j}(u_i - u_j)(v_i - v_j)}{\prod_{i,j}(u_i + v_j)} = \sum_{\sigma} (-1)^{\sigma} \prod_i \frac{1}{u_i + v_{\sigma(i)}},
\]

**1/cosh is Fourier-invariant!**

\[
\frac{1}{\cosh \pi p} = \int dx \frac{e^{2\pi ipx}}{\cosh \pi x} \leftrightarrow \frac{1}{2 \cosh p} = \frac{1}{\pi} \int dx \frac{e^{2i\pi px}}{2 \cosh x}
\]
How to calculate the Partition function (1)

\[
Z(N, k) = C_{N,k} g(N, k)
\]

\[
C_{N,k} = \frac{1}{(4\pi k)^N \cdot N!}
\]

\[
g(N, k) = \int d^N x \frac{\prod_{i<j} \tanh^2((x_i - x_j)/2k)}{\prod_i 2 \cosh(x_i/2)}
\]
How to calculate the Partition function (2)

\[ e^{-S(N,k;x)} = \prod_{i<j} \tanh^2((x_i - x_j)/2k) \]

\[ \frac{g(N, k_2)}{g(N, k_1)} = \frac{\int d^N x e^{-S(N,k_2;x)}}{\int d^N x e^{-S(N,k_1;x)}} = \left\langle e^{-S(N,k_2;x)+S(N,k_1;x)} \right\rangle_{N,k_1} \]

calculable by Monte Carlo, when \( k_1 \) and \( k_2 \) are close.

(rem: the integral is well-defined for non-integer \( k \) as well)
How to calculate the Partition function (3)

\[ F = -\log C_{N,k} - \sum_{i=1}^{l} \log \left( \frac{g(N, k_i)}{g(N, k_{i-1})} \right) - \log g(N, 0) \]

\[ g(N, 0) = \int \frac{d^N x}{\prod_i 2 \cosh(x_i/2)} = \pi^N \]

\[ (0 = k_0 < k_1 < \cdots < k_l = k) \]
How to calculate the Partition function (4)

<table>
<thead>
<tr>
<th>$N$</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$g(N, k_2)/g(N, k_1)$</th>
<th>$-\log[g(N, k_2)/g(N, k_1)]$</th>
<th>$-\log[g(N, k_2)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>1.0</td>
<td>0.636057 ± 0.0005008</td>
<td>0.452467097 ± 0.000787350819</td>
<td>-1.83699267 ± 0.000787350819</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>2.0</td>
<td>0.636303 ± 0.0003078</td>
<td>0.452080414 ± 0.000483731807</td>
<td>-1.38491226 ± 0.00127108263</td>
</tr>
<tr>
<td>2</td>
<td>2.0</td>
<td>3.0</td>
<td>0.667156 ± 0.0001956</td>
<td>0.404731377 ± 0.000293184802</td>
<td>-0.980180884 ± 0.00156426743</td>
</tr>
<tr>
<td>2</td>
<td>3.0</td>
<td>4.0</td>
<td>0.700369 ± 0.0001318</td>
<td>0.35614794 ± 0.000188186513</td>
<td>-0.624032944 ± 0.00175245394</td>
</tr>
<tr>
<td>2</td>
<td>4.0</td>
<td>5.0</td>
<td>0.730581 ± 0.00009278</td>
<td>0.313915171 ± 0.000126994816</td>
<td>-0.310117773 ± 0.00187944876</td>
</tr>
<tr>
<td>2</td>
<td>5.0</td>
<td>6.0</td>
<td>0.756683 ± 0.00006737</td>
<td>0.278810872 ± 0.000089033204</td>
<td>-0.0313069007 ± 0.00196848208</td>
</tr>
<tr>
<td>2</td>
<td>6.0</td>
<td>7.0</td>
<td>0.779218 ± 0.0000505</td>
<td>0.249464426 ± 0.0000648085645</td>
<td>0.218157525 ± 0.00203329064</td>
</tr>
<tr>
<td>2</td>
<td>7.0</td>
<td>8.0</td>
<td>0.798323 ± 0.00003836</td>
<td>0.225242002 ± 0.0000480507263</td>
<td>0.443399527 ± 0.00208134137</td>
</tr>
<tr>
<td>2</td>
<td>8.0</td>
<td>9.0</td>
<td>0.814825 ± 0.00003</td>
<td>0.204781913 ± 0.0000368177216</td>
<td>0.64818144 ± 0.00211815909</td>
</tr>
<tr>
<td>2</td>
<td>9.0</td>
<td>10.0</td>
<td>0.828998 ± 0.0000237</td>
<td>0.187537536 ± 0.00002858873</td>
<td>0.835718976 ± 0.00214674782</td>
</tr>
</tbody>
</table>
Free energy of $U(2)_k \times U(2)_{-k}$ theory is known. (Okuyama 2011)

Good agreement! (→our method works indeed)
let’s test Fuji-Hirano-Moriyama formula!

looks fine, but...
there is a small discrepancy...

the discrepancy is almost N-independent and hence is not a worldsheet instanton effect
• The discrepancy turns out to be kind of integration constants, which have been missed in the analysis.

• M. Marino taught us the analytic form of this term around the planar limit. ("constant map")

• Correction emerges at each order in $1/k^2$ ($= g_{\text{string}}$) → it must be taken into account when one compares it with the string theory.

• "Fuji-Hirano-Moriyama + worldsheet instanton + constant map" seems OK at any $N$ and any $k$.

• Disagreement with string prediction, at least naively. → have to modify the dictionary??
Planar limit ($\lambda=N/k$ fixed)

$\partial_\lambda F_0(\lambda)$

Integrating constant does not matter

\[ \text{exact} \]

\[ \text{1-loop} \] (perturbation in the gauge theory)

SUGRA

(from M. Marino's talk in strings2011)

IIA SUGRA is reproduced!
Large-N, fixed $k$

- The integration constants give only subleading contribution. $\rightarrow$ The $N^{3/2}$ law holds indeed.

$11d$ SUGRA has been reproduced!

\[ F \sim \frac{\pi \sqrt{2}}{3} k^{1/2} N^{3/2} \]
Wilson loop
**1/6 BPS Wilson loop**

\[ W_{1/6,R} = \frac{1}{d_R} Tr P \exp \left( \int ds \left( i A_\mu \dot{x}^\mu + \frac{2\pi}{k} |\dot{x}| M_{IJ} \Phi^I \bar{\Phi}^J \right) \right) \]

- **circular** → keeps 1/6 of SUSY

**fundamental representation**

\[ \langle W_{1/6,\text{fund}} \rangle = \frac{1}{N} \sum_i \langle e^{\mu_i} \rangle_{M.M.} \]

- **sign problem** → wtf

\[ \langle W_{1/6} \rangle = \frac{2Ne^{\frac{\pi i}{k}}}{\cos \frac{\pi}{k}} \left\langle e^{-\frac{x_1}{k}} \prod_{j=2}^N \left[ \frac{\tanh \left( \frac{x_1 - x_j - 2\pi i}{2k} \right)}{\tanh \left( \frac{x_1 - x_j}{2k} \right)} \right] \right\rangle \]

- **sign-free theory** → easy
phase factor disagrees with analytic calculation
(Drukker-Marino-Putrov 2010) at planar & strong coupling
→ typo in DMP! (thanks to M. Marino)
Summary

- ABJM partition function and Wilson loops can be calculated numerically at finite N and finite coupling.
- Corrections to previous analytic calculations are found.
- The full result of the free energy does not seem to allow a nice analytic expression.
Outlook

• Test of AdS/CFT at quantum string level
  
  disagreement between Fuji-Hirano-Moriyama and Bergman-Hirano
  
  need modification(s) in gauge and/or gravity side?

• Wilson loops in symmetric/antisymmetric representation → comparison to D-brane world-volume

• Other theories, topological string,...
Summary of the lectures
• SUSY can be studied on computer, by combining lattice and non-lattice methods (e.g. Matrix model).

• Simulation of the quantum gravity → inflation, birth of the universe, multiverse, Hawking evaporation...

• Not string theorists, but lattice theorists, can study such exciting topics.
THE END